Cheating at coin tossing

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Kent workshop, Paris, 9-10 June 2011

Randomness vs. determinism

Is coin tossing

- **random**? (definit probability?)
- or **deterministic**? (control?)
- Is this an exclusion?

Outline

Arguments for

- A "fair" coin has probability 1/2.
- There is no physical probability attached to the coin, we can cheat on each toss (by sufficient control).

My aim:

- The coin toss is **fine-grained deterministic**, but **coarsgrained random**.
- This explains why certain coin tossings allow to cheat (control is fine-grained), certain won't (control is coars-grained).
- Apparent randomness depends on two parameters: the uncertainty in control and the quasi-chaotic dynamics of the coin.

Deterministic vs. Random

A system $(\Gamma, (\phi_t)_{t\in\mathbb{R}})$ is **deterministic**, if $\phi_t : \Gamma \to \Gamma$ is a function.

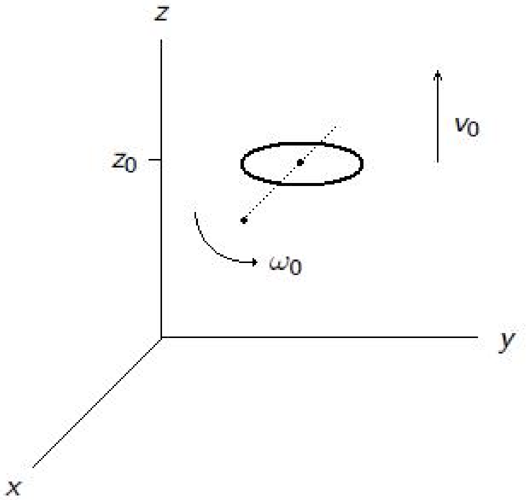
A system $\phi_{(t)} : \Gamma \to \Theta$, with Θ partitionned into two possible outcomes $\{A, \neg A\}$ is **fine grained deterministic**, if there exists a function

$$\chi_A : \Gamma \to \{0, 1\}$$

A system $\phi_{(t)} : \Gamma \to \Theta$ is $\epsilon-$**coarse grained random**, if there exists $\epsilon > 0$ such that for every $p \in \Gamma$ and every open ball $B_\delta(p)$ $(\delta \geq \epsilon)$, A has a non trivial probability in that ball, i.e.

$$P_{\delta,p}(A) := P(A|B_\delta(p)) \notin \{0, 1\}$$

Keller coin



The Keller coin is fine grained deterministic

A coin, tossed with initial velocity v_0 at hight z_0, will, at t, be at hight

$$z(t) = z_0 + v_0 t - (g/2) t^2$$

Elapsed time until return to z_0:

$$t^* = 2v_0/g$$

Flips per second

$$n_0 = \omega_0/\pi, \quad \omega_0 \text{ angular velocity}$$

Number of **Flips**

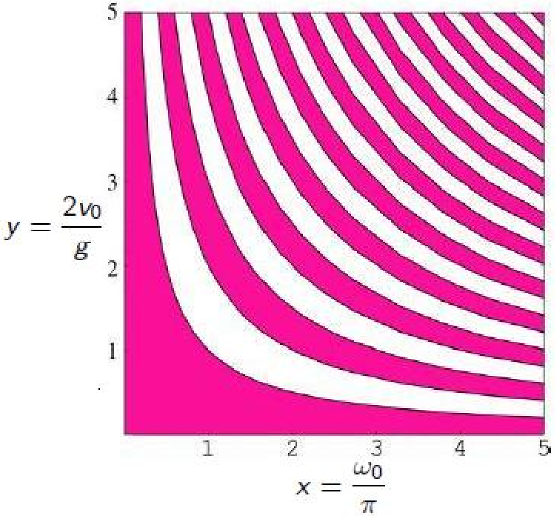
$$n = n_0 t^* = \frac{\omega_0}{\pi}\frac{2v_0}{g}$$

Coin lands

$$\begin{array}{lll} \text{same side up} & \text{if} & 0 \le n \le 1 \mod 2 \\ \text{other side up} & \text{if} & 1 \le n \le 2 \mod 2 \end{array}$$

But also coars-grained random

Hyperbolas defined by $j = xy$, $j \in \mathbb{N}$



c.f. Keller 1986

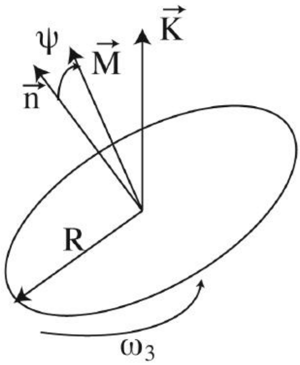
For physical probability

Assume a density on the (x, y) space. If it is approximately constant within a distance corresponding to two ribbons then

$$P(H) \approx P(T) \approx \frac{1}{2}$$

However, this becomes less true the smaller the control-ball.

Cheating at the coin



cf. Diaconis (2007).

Control $\psi \leq \pi/4$, to cheat.

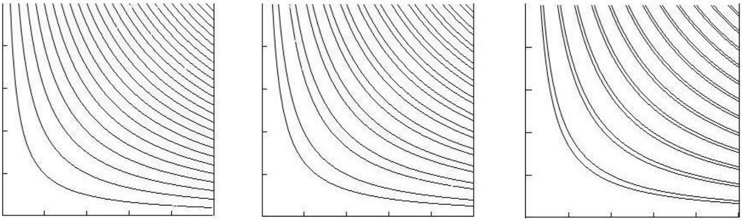
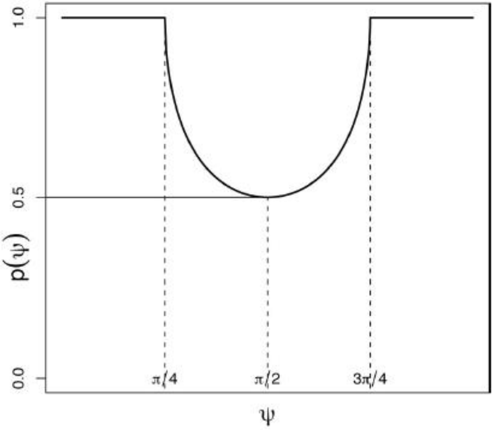


Figure: Coin with $\psi = \frac{\pi}{2}$, $\psi = \frac{5}{16}\pi$, $\psi = \frac{26}{100}\pi$.

Hyperbolas defined by

$$xy = \begin{cases} j & \text{if} \quad j = 2n \\ j + \frac{2}{\pi}\sin^{-}\cot^2\psi & \text{else} \end{cases}$$

with $y = \frac{2v_0}{g}$, $x = \frac{\omega_M}{\pi}$.



Diaconis (2007)

$$P_\psi = \begin{cases} \frac{1}{2} + \frac{\sin^{-} \cot^2 \psi}{\pi} & \text{if} \quad \frac{\pi}{4} \leq \psi \leq \frac{3\pi}{4} \\ \frac{1}{2} & \text{else} \end{cases}$$

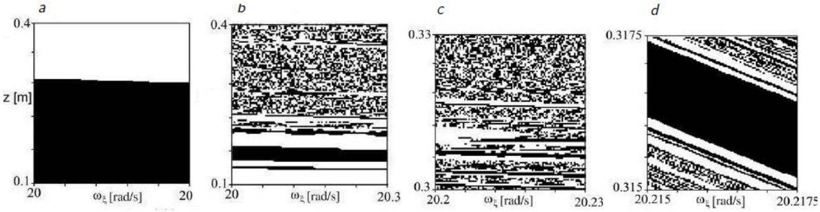
Against physical probability

There is no physical probability of the coin (cf. Jaynes 2003). Everything depends on controling the tossing.

- Are there systems where, tossing control might not be enough to cheat?
- This has to do with the shape of the bassins of attractions which has to do with the sensitivity of the mechanism.

Bouncing (after free fall) with precession

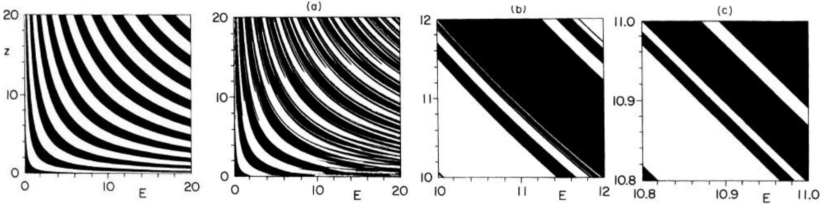
ω_ξ rotation around parallel to x axis.



Strzalko (2008)

Figure: (a) Keller coin (b) bouncing coin with successive enlargements (c, d) (no air resistance)

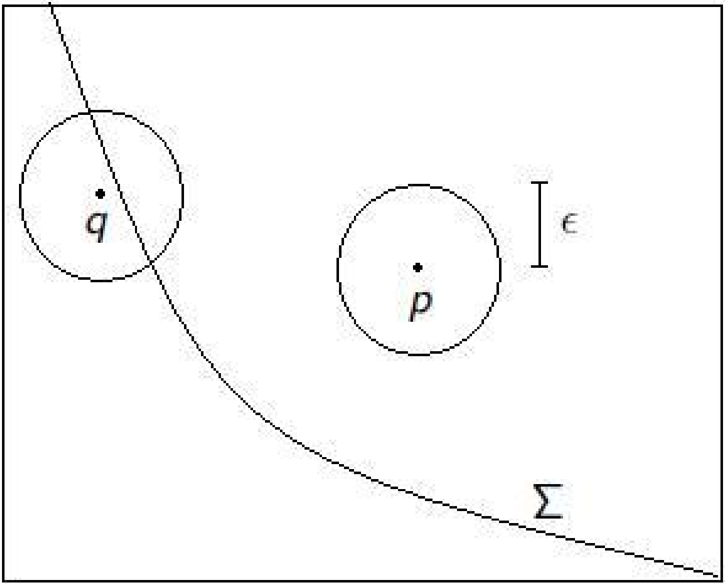
… without Precession

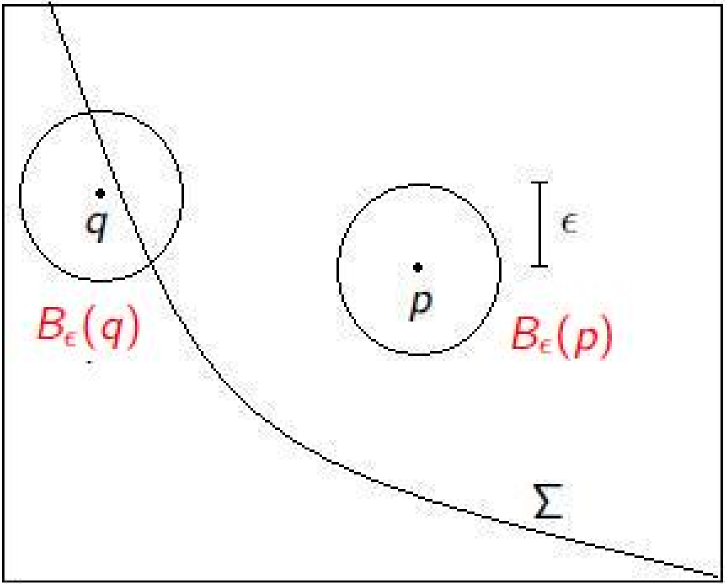


Vulovic, Prange (1986)

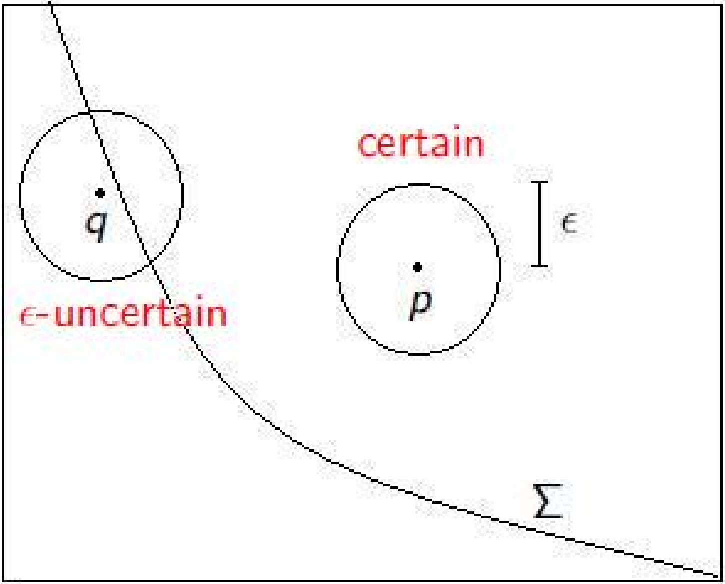
Figure: Keller coin and (a) bouncing coin with successive enlargements (b,c). $E = 0.51\omega^2$.

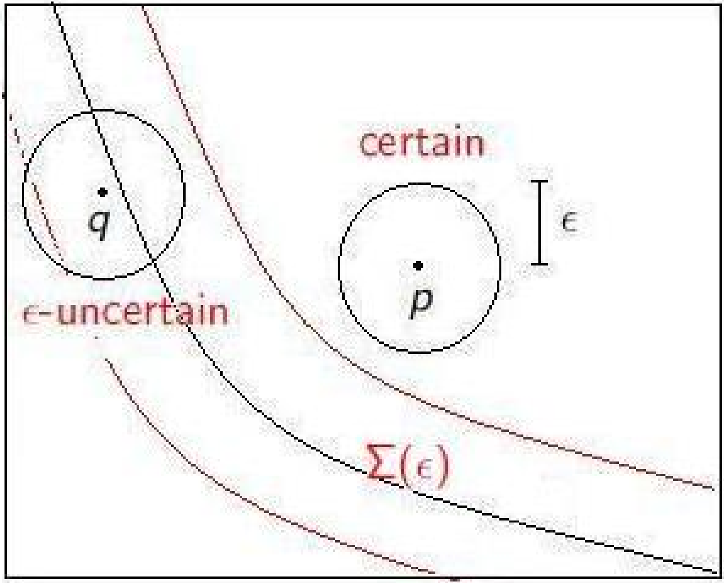
Probability of error

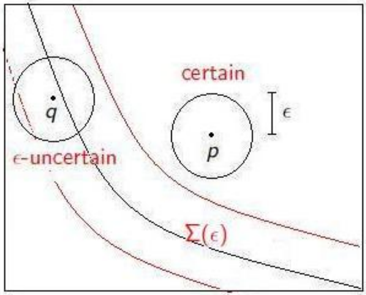




q
$B_\epsilon(q)$
p
$B_\epsilon(p)$
ϵ
Σ


certain
ϵ
q
p
ϵ-uncertain
Σ


certain
ϵ
q
p
ϵ-uncertain
$\Sigma(\epsilon)$



The probability of error

$$f(\epsilon) = \frac{\mu(\Sigma_\epsilon)}{\mu(S)}$$

can be interpreted as a **measure of how probable it is that cheating fails** (given the error ϵ in control).

If Σ is **non fractal** then

$$f(\epsilon) \sim \epsilon$$

If Σ is **fractal** then

$$f(\epsilon) \sim \epsilon^{\alpha}$$

$$\lim_{\epsilon} \frac{\ln f(\epsilon)}{\ln \epsilon} = \alpha$$

$$\alpha = N - D_0, \qquad \alpha < 1$$

Eg. $\alpha = 0.1$ To reduce $f(\epsilon)$ by a factor 10 we need to reduce ϵ by a factor 10^{10}.
Improvent in prediction by improving accuracy in IC becomes harder as $\alpha \to 0$.

Variation of f_ϵ

f_ϵ might vary in phase space (with location p)

$$f_\epsilon(p) = \frac{\mu(\Sigma_\epsilon(p))}{\mu(B_\epsilon(p))}$$

Although Σ is not fractal, $\Sigma_\epsilon(p)$ can "appraoch fractality" as $p \longrightarrow \infty$.
Then

$$f_\epsilon(p) \sim \epsilon^\alpha$$

for $p \longrightarrow \infty$.

Conclusion

If a system is fine-grained deterministic and coarse grained random, then

- The probability of error depends not only on our general ability to control, but also on the regions of (high/low) sensitivity of the system.
- One may argue for a certain definit probability of an outcome in a system, if across different regions the probability of error is high and the color pattern is sufficiently regular.

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