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Newton said "I seem to have been only like a boy playing on the sea s
and diverting myself in now and then finding a smoother pebble or
prettier shell than ordinary, whilst the great ocean of truth lay a
undiscovered before me."

The aim of these lectures is to show some of the beautiful peb
and shells I know, which are discrete Painlevé equations.

## §1 Introduction

In 1939, Shohat (Duke Math. J. 5 (2) (1939) 401-417) extended the study of orthogonal polynomials $\{\Phi_n(x)\}_{n=0}^{\infty}$ to the class with weight $p(x)$

$$\int_a^b p(x)\,\Phi_m(x)\,\Phi_n(x)\,dx = 0, \qquad m\neq n,\ m,n\in\mathbb{N}$$

s.t.

$$p(x) = \frac{1}{A(x)}\exp\left(\int_a^b \frac{B(x)}{A(x)}\,dx\right), \qquad \text{with } A\neq 0 \text{ in } (a,$$

In the case $a=-\infty$, $b=\infty$, $p(x)=\exp(-x^4/4)$, he obtained

$$\lambda_{n+2}\left(\lambda_{n+1}+\lambda_{n+2}+\lambda_{n+3}\right) = n+1 \qquad (1.1)$$

where $\lambda_n$ is related to the 3-term recurrence relation

$$\Phi_n(x) - (x-c_n)\,\Phi_{n-1}(x) + \lambda_n\,\Phi_{n-2}(x) = 0, \quad n\geq 2$$

$$\Phi_0 = 1,\ \Phi_1 = x-$$

Eqn (1.1) is a special case of what is now called the first disc
Painlevé equation $(dP_1)$:

$$w_n(w_{n+1} + w_n + w_{n-1}) = \alpha n + \beta + \delta(-1)^n + \gamma w_n \qquad (1\text{-}2)$$

where $\alpha, \beta, \gamma, \delta$ are consts.

The name comes from its continuum limit. Take $\delta = 0$ and write

$$t = \varepsilon n, \quad w_n = \mu + \varepsilon^2 u(t), \quad u(t) \text{ assumed analyt}$$

$$\alpha = -\varepsilon^a \nu, \quad \beta = -\frac{\gamma^2}{12}$$

then

$$w_{n\pm 1} = \mu + \varepsilon^2 u(t \pm \varepsilon)$$

$$= \mu + \varepsilon^2 \left\{ u(t) \pm \varepsilon u_t(t) + \frac{\varepsilon^2}{2} u_{tt}(t) + O(\varepsilon^3) \right\}$$

So we get from (1-2):

$$(\mu + \varepsilon^2 u)\left(3\mu + 3\varepsilon^2 u + \varepsilon^4 u_{tt} + O(\varepsilon^6)\right)$$

$$= -\varepsilon^{a-1} \nu t - \frac{\gamma^2}{12} + \gamma\mu + \gamma \varepsilon^2 u$$

Take

$$3\mu^2 = \gamma\mu - \frac{\gamma^2}{12} \qquad 3\mu^2 = \gamma^2/6 - \gamma^2/12 = \gamma^2/12$$

$$6\mu = \gamma \qquad \Rightarrow \quad \mu = \gamma/6$$

ID

and

$$a - 1 = 4$$

$\Rightarrow$

$$\mu u_{tt} + 3u^2 = -\nu t + O(\varepsilon^2)$$

Taking $\mu = -\frac{1}{2}$, $\nu = -\mu$, we get as $\varepsilon \to 0$

$$u_{tt} = 6u^2 + t$$

the first Painlevé eqn $P_I$, one of six classical Painlevé e[qns] denoted $P_I - P_{VI}$.

Equation (1-2) is not the only discrete version of $P_I$.
For example: using the notation $w = w_n$, $\overline{w} = w_{n+1}$, $\underline{w} = w_{n-1}$,

- $$\frac{\bar{z}}{\overline{w}+w} + \frac{z}{w+\underline{w}} + 2w^2 + t = 0$$

the "alternative $dP_I$"
where $z$ is linear in $n$ & $(-1)$

- $$w_{n+1}\, w_{n-1} = \frac{w_n - t_0 q^n}{w_n^2}, \quad q \neq 0, 1 \qquad (1\text{-}5)$$

$q$-discrete $P_I$ or $qP_I$
where $t_0$, $q$ are consts

are both integrable discrete versions of $P_I$.

The case $\delta \neq 0$ of Eqn (1-2) has to be converted to a syst
before attempting a continuum limit, i.e. define

$w_{2k} = u_k$

$w_{2k+1} = v_k$

Case $n = 2k$: $\quad u_k(v_k + u_k + v_{k-1}) = \dfrac{2\alpha k + \beta + \delta}{u_k} + \gamma \qquad (1\text{-}3$

Case $n = 2k+1$: $\quad v_k(u_{k+1} + v_k + u_k) = \dfrac{(2\alpha k + \alpha + \beta - \delta)}{v_k} + \gamma \qquad (1\text{-}$

The system (1-3)–(1-4) is often called the "asymmetric" $dP_I$.
(In Japan, it is called $dP_2$.)

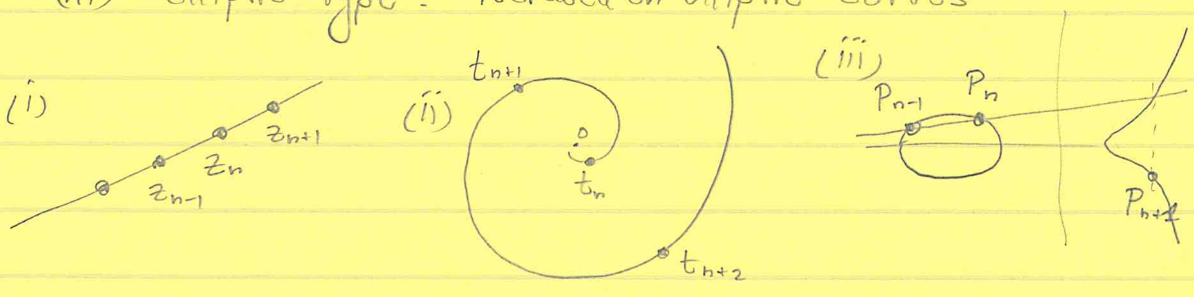
Exercise: show that (1-3)–(1-4) has a continuum limit to
$$y_{tt} = 2y^3 + ty + a, \quad a \text{ is const}$$
which is $P_{II}$.

In general, $\exists$ three-types of discrete Painlevé equations (DPEs)

(i) additive-type : iterated on a straight line $z_n = \alpha n + \beta$ like (1-3)-(1-4)

(ii) multiplicative- or q-type: iterated on a spiral $t_n = t_0 q^n$ like (1-5)

(iii) elliptic-type : iterated on elliptic curves




$(|q|>1)$

Type (i) DPEs arise as recurrence relations from transformations of ODEs. Consider $P_{IV}(w; t; a, b)$:

$$W'' = \frac{W'^2}{2W} + \frac{3W^3}{2} + 4tW^2 + 2(t^2 - a)W - \frac{b^2}{2W} \qquad (1\text{-}6)$$

Which has Bäcklund transformations : mapping

$$P_{IV}(w_n, t; a_n, b_n) \longmapsto P_{IV}(w_{n\pm 1}, t; a_{n\pm 1}, b_{n\pm 1})$$

with

$$a_n = -\frac{n}{2} - \frac{c_0}{2} + \frac{3c_1}{2}(-1)^n$$

$$b_n = n + c_0 + c_1(-1)^n \qquad c_0, c_1 \text{ consts.}$$

$$2w_n w_{n+1} = -w_n' - w_n^2 - 2t w_n + b_n \qquad (1.7a)$$

$$2w_n w_{n-1} = w_n' - w_n^2 - 2t w_n + b_n \qquad (1.7b)$$

Adding, we get

$$2w_n (w_{n+1} + w_n + w_{n-1}) = -4t w_n + 2b_n$$

which is (1-2) with $\gamma = -2t$, $\alpha = 1$, $\beta = c_0$, $\delta = c_1$.

Such transformations are simpler to understand in terms of reflections.
To see this, consider the symmetric form of $P_{IV}$:

$$\dot{f}_0 = f_0(f_1 - f_2) + \alpha_0, \quad f_j = f_j(s)$$
$$\dot{f}_1 = f_1(f_2 - f_0) + \alpha_1,$$
$$\dot{f}_2 = f_2(f_0 - f_1) + \alpha_2,$$

where $f_0 + f_1 + f_2 = s$, $\alpha_0 + \alpha_1 + \alpha_2 = 1$.

Exercise: Eliminate $f_1, f_2$ and show that $f_0(s) = \mu\, w(t)$ with $t = \lambda s$ satisfies $P_{IV}$ (eqn (1-6)) for some $\lambda, \mu, \alpha, b$

The BT's can be written in terms of the operations:

$$s_i(\alpha_i) = -\alpha_i \qquad s_i(\alpha_j) = \alpha_j + \alpha_i, \quad j = i \pm 1, \quad \pi(\alpha_j) = \alpha_{j+1}$$
$$s_i(f_i) = f_i \qquad s_i(f_j) = f_j \pm \frac{\alpha_i}{f_i}, \quad j = i \pm 1, \quad \pi(f_j) = f_{j+1}$$
$$j \in \mathbb{N} \pmod 3.$$

See Noumi & Yamada (1999)
Okamoto (1986)

For example, the BT (1-7a) arises if we take

$$T = \pi \circ s_2 \circ s_1$$

Check
$$\begin{aligned} T(\alpha_0) &= \pi \circ s_2(s_1(\alpha_0)) \\ &= \pi \circ s_2(\alpha_1 + \alpha_0) \\ &= \pi(s_2(1 - \alpha_2)) \\ &= \pi(1 + \alpha_2) \\ &= 1 + \alpha_0 \end{aligned}$$

Exercise: See also Tutorial 1, Q 1. Show that $T(f_0)$ is related to the BTs (1-7).

The operations $s_j, \pi$ satisfy $s_j^2 = 1$, $(s_j s_{j+1})^3 = 1$, $\pi^3 = 1$,
$\pi s_j = s_{j+1} \pi$.

These properties make the span

$$W = \langle s_0, s_1, s_2 \rangle$$ an affine Weyl (Coxeter) group or w... system,

and

$$\widetilde{W} = \langle s_0, s_1, s_2, \pi \rangle$$ an extended affine Weyl group

The root system associated with $P_{IV}$ is $A_2^{(1)}$. (We explore this further in Lecture 2.)

Sakai (2001) showed ∃ 22 classes of DPEs described by root sys...
⟺ initial value spaces

| Type of iteration | Type of Initial Value Space |
|---|---|
| Elliptic | $A_0^{(1)}$ |
| Multiplicative | $A_0^{(1)*}, A_1^{(1)}, \ldots, A_8^{(1)}, A_7^{(1)'}$ |
| Elliptic | $A_0^{(1)**}, A_1^{(1)*}, A_2^{(1)*}, D_4^{(1)}, \ldots, D_8^{(1)}, E_6^{(1)}, \ldots, E_8^{(1)}$ |

Each equation also has a symmetry group. To be a DPE, time itera... has to be a tranlation on the root lattice.

In these lectures, I will show how to construct
- the initial value space
- the symmetry group

of a given DPE and, conversely, given a parametrization of the initial value space, how to construct the corresponding DPE. I will focus on examples.

DPEs are pretty shells – but a lot can be understood about their solutions by understanding Sakai's theory.