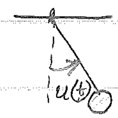
LMS OPSFA Summer School – Nalini Joshi

## L1 §2 Initial Value Spaces

Initial value spaces are like phase spaces of mechanical systems.

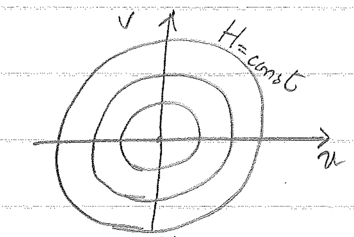



Consider a simple pendulum, moving under influence of gravity:

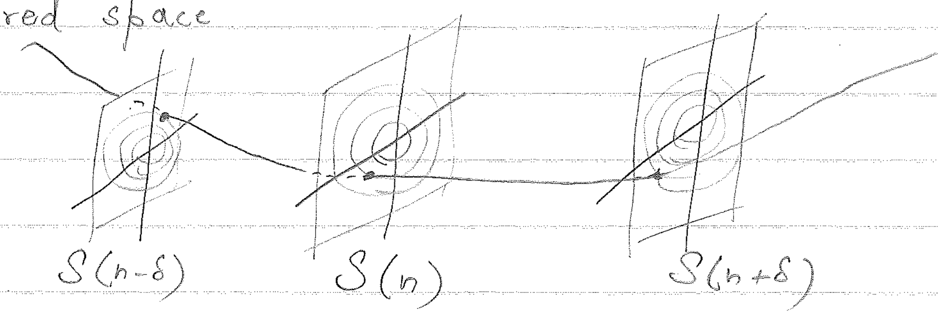
$$\begin{cases} \dot{u}(t) = v(t) \\ \dot{v}(t) = -u(t) \end{cases} \Rightarrow H = \frac{v^2}{2} + \frac{u^2}{2} \qquad (2.1)$$

is conserved in time.

Knowledge of initial values $(u(0), v(0))$ determines $H$ for all $t$, and therefore describes the motion qualitatively for all time.




For DPEs, we will see that solutions follow trajectories in a fibred space




The plane pendulum motion is actually governed by

$$\begin{cases} \dot{u}(t) = v(t) \\ \dot{v}(t) = -\sin(u(t)) \end{cases} \qquad (2.2)$$

where the case $|u| \ll 1$ gives the limiting form (2.1).

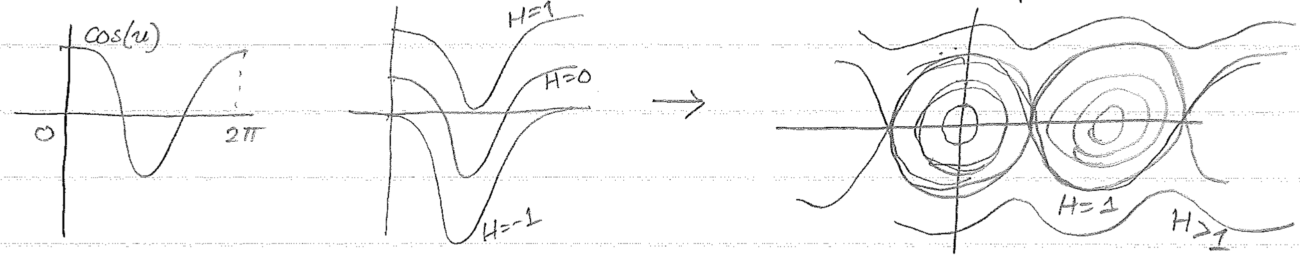
This system also has conserved total energy

$$H(u,v) = \frac{v^2}{2} - \cos(u)$$

Now the motion parametrizes curves that are no longer circles.

$u(0) = \frac{(2n+1)\pi}{2}$, $v(0) = 0 \Rightarrow H = 0 \Rightarrow v = \pm\sqrt{2\cos(u)}$

$u(0) = \pi$, $v(0) = 0 \Rightarrow H = 1 \Rightarrow v = \pm\sqrt{2(1+\cos(u))}$




Consider the plane pendulum in complex variables. Let

$$x(t) = \exp(iu(t))$$

$$\Rightarrow \quad \dot{x} = iv\cdot x \quad \Rightarrow \quad v = -i\frac{\dot{x}}{x}$$

$$\Rightarrow \quad H = -\frac{\dot{x}^2}{x^2} - \frac{1}{2}\left(x + \frac{1}{x}\right)$$

or writing $\dot{x} = y$, we obtain

$$f(x,y) = y^2 + kx^2 + \frac{1}{2}(x^3 + x) = 0, \quad \text{where } H = k.$$

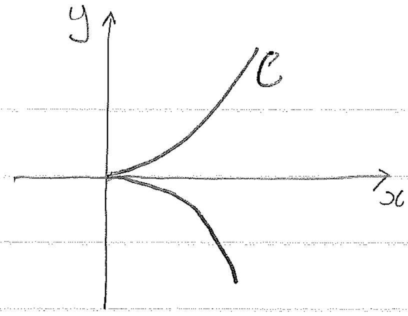
Note that all these curves intersect at $(x,y) = (0,0)$, regardless of the value of $H=k$. This is called a base point.

This causes difficulties in the description of the motion $(x(t), y(t))$ of points on such curves. How do we distinguish the motion and continue the solutions through such points.

To explain the procedure that answers this question, we consider a simpler curve.

$$C: \quad f(x,y) = y^2 - x^3 = 0,$$

which has a singularity at $(0,0)$. I.e. both $f$ and $\nabla f$ vanish there. Note: $\nabla f = (f_x, f_y) = (-2x^2, 2y)$.




We can resolve the flaw of the parametrization of the curve through $(0,0)$ by taking a sequence of new coordinates. The first transformation is

$$(x, y) = (x_1, x_1 y_1) \Rightarrow x_1 = x, \; y_1 = y/x$$

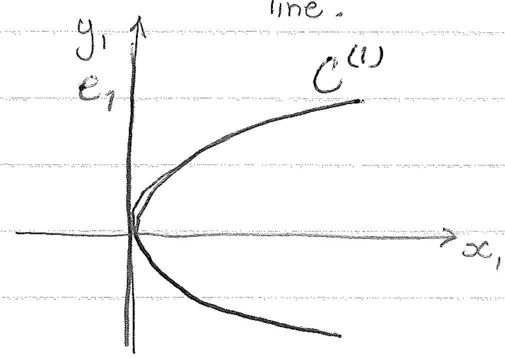
$\Rightarrow$

$$f(x,y) = x_1^2 y_1^2 - x_1^3 = \underbrace{x_1^2}_{e_1 = \{x_1 = 0\}} \underbrace{(y_1^2 - x_1)}_{f^{(1)} = y_1^2 - x_1}$$

$e_1 = \{x_1 = 0\}$ called an exceptional line.

$f^{(1)} = y_1^2 - x_1$ called the strict transform of $f$.

$\Rightarrow$



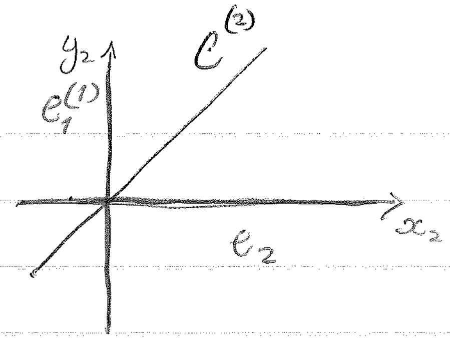

where $C^{(1)}: f^{(1)}(x_1, y_1) = 0$.

But $e_1$ and $C^{(1)}$ being tangential at $(0,0)$ still causes ambiguity for the parametrizing functions. So we resolve again:

$$(x_1, y_1) = (x_2 y_2, y_2) \Rightarrow x_2 = \frac{x_1}{y_1}, \; y_2 = y_1$$

$$\Rightarrow f^{(1)}(x_1, y_1) = y_2^2 - x_2 y_2 = \underbrace{y_2}_{e_2 = \{y_2 = 0\}} \underbrace{(y_2 - x_2)}_{f^{(2)} = y_2 - x_2}$$

and $e_1$ becomes $\{x_2 = 0\} = e_1^{(1)}$.




But $C^{(2)}$, $e_1^{(1)}$ and $e_2$ all intersect at $(x_2, y_2) = (0,0)$. This still causes a problem. (Note that $e_1$ is now $e_1 - e_2$ and no longer contains this point.)

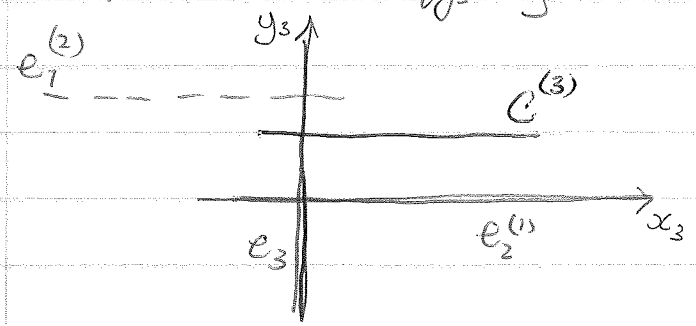
So we resolve again:

$$(x_2, y_2) = (x_3, x_3 y_3) \quad \text{or} \quad (x_3, y_3) = (x_2, y_2/x_2)$$

$\Rightarrow$

$$f^{(2)}(x_2, y_2) = x_3 y_3 - x_3 = \underbrace{x_3}_{e_3 : \{x_3 = 0\}} \underbrace{(y_3 - 1)}_{f^{(3)}}$$

Note: $e_2^{(1)}$ is $\{y_3 = 0\}$ but $e_1^{(1)}$ becomes $y_3 = \infty$, no longer visible.




Now the curves only intersect pairwise and do so transversely. This is known as a "good resolution," and is guaranteed to exist by Hironaka's theorem in a field of any characteristic.

Each step above is called a "blow-up" and such operations were well known to Newton, who was studying Puiseux series expansions of $y$ as a function of $x$ around such singularities.

The same approach works to resolve a pencil of curves through its base points.

Note that any curve obtained as an exceptional line from a blow-up is a curve of self-intersection number that is one less than the curve we started from.

So $e_1$ has self-intersection $-1$

$e_1 - e_2$ " " $-2$,

and $e_2 - e_3$ " " $-2$.

Behind the space we have constructed lies a deeper algebraic structure. Consider the lines of self-intersection $-2$:

$$\alpha_1 = e_1 - e_2$$
$$\alpha_2 = e_2 - e_3$$

and equip the equivalence classes of all lines with an intersection form $(\ |\ )$:

$$(e_i | e_j) = \begin{cases} -1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Note: $(\alpha_i | \alpha_i) = -2 \quad , i = 1,2$

and $(\alpha_1 | \alpha_2) = +1$

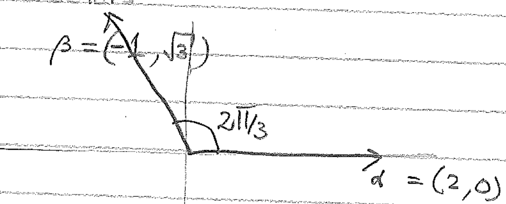
$\Big\} \Rightarrow 4\cos(\theta_{12}) = 1$ when interpreted as an inner product, with $\theta_{12}$ as angle between $\alpha_1, \alpha_2$.

This leads to a reflection group of type $A_2$.

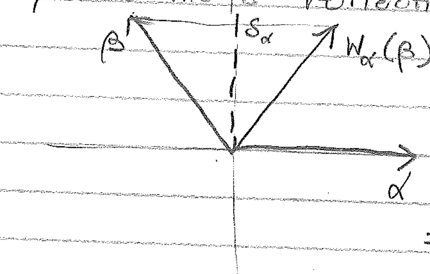
Every algebraic curve corresponds to a reflection group of type A-D-E. (This is known as Du Val or McKay correspondence, and Arnol'd's classification of curves.)

When the "curves" are no-longer autonomous, we obtain a classification of initial value spaces of Painlevé and discrete Painlevé equations.

L2 Consider two vectors

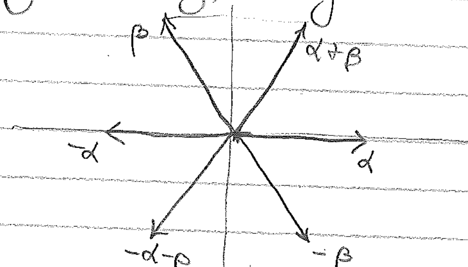



Reflecting $\beta$ in $\alpha$ means reflecting it across a mirror orthogonal to $\alpha$



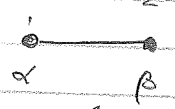

$$W_\alpha(\beta) = \beta - \frac{2(\alpha,\beta)\alpha}{(\alpha,\alpha)}$$
$$= (-1, \sqrt{3}) - \frac{2(-2)}{4}\cdot(+2,0)$$
$$= (1, \sqrt{3})$$
$$= \alpha + \beta$$

Repeatedly reflecting, we get




a "root system" with 6 roots. It forms a group under reflection. Every element is an integer linear combination of $\alpha, \beta \Rightarrow$ these are "simple roots".

The group is named $A_2$ and represented by the Dynkin diagram




More generally, a reflection group or Weyl group has

- a root system $\alpha_1, \dots, \alpha_n$
- reflections $W_i(\alpha_j) = \alpha_j - 2\dfrac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}\alpha_i$
- co-roots $\alpha_i^\vee = \dfrac{2\alpha_i}{(\alpha_i, \alpha_i)}$
- weights $h_1, \dots, h_n$
  s.t. $(h_i, \alpha_j^\vee) = \delta_{ij}$

Satisfying the crystallographic property : $(\alpha_i, \alpha_j^\vee) \in \mathbb{Z}$

$\Rightarrow (\alpha_i, \alpha_j^\vee)(\alpha_i, \alpha_j^\vee) = 4\cos^2(\theta_{ij}) \in \mathbb{N}$

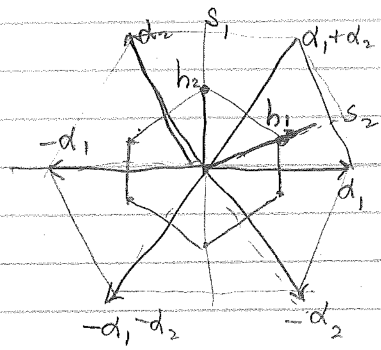
$\Rightarrow \cos^2(\theta_{ij}) = \dfrac{n}{4}, \; n \in \mathbb{N}$

i.e. $\cos^2(\theta_{ij}) = 0, \frac{1}{4}, \frac{1}{2}$ or $\frac{3}{4}$

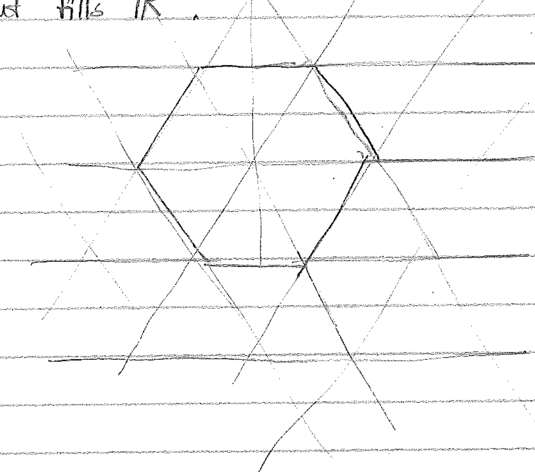
$$\Rightarrow \quad \cos(\theta_{ij}) = 0, \pm\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, \pm\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \theta_{ij} = \pi - \theta_{s_i s_j} = \pi - \frac{\pi}{m_{ij}}$$ , with $m_{ij} = 2, 3, 4$ or $6$.

For $A_2$:




Now take translations of the root system to build an affine Weyl lattice that fills $\mathbb{R}^2$.

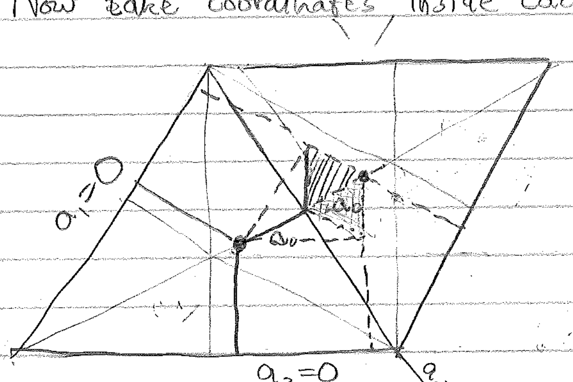


finite gp

$A_2 \Rightarrow$ a triangular lattice $A_2^{(1)}$ affine reflections on the lattice give us an affine Weyl group also called $A_2^{(1)}$.

infinite gp

Now take coordinates inside each triangle given by $(a_0, a_1, a_2)$.




Let $s_j$ be the reflection across the side given by $a_j = k$, for some $k \in \mathbb{Z}$.

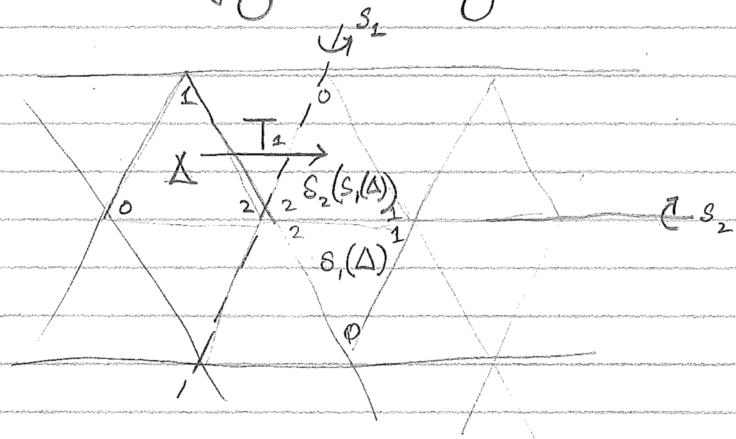
Note: $a_0 + a_1 + a_2 =$ const $= k$ say.
The shaded triangle is isosceles.

$$\Rightarrow \quad s_0(a_0, a_1, a_2) = (-a_0, a_1 + a_0, a_2 + a_0)$$

Similarly

$$s_1(a_0, a_1, a_2) = (a_0 + a_1, -a_1, a_2 + a_1)$$
$$s_2(a_0, a_1, a_2) = (a_0 + a_2, a_1 + a_2, -a_2)$$

We want to develop dynamics on this lattice by taking a translation from one triangle to a conjugate triangle.




We need a permutation $\pi$ to get back the original labelling.

$\Rightarrow$

$T_1 = \pi \circ s_2 \circ s_1$ where $\pi(0,1,2) = (1,2,0)$.

Similarly $T_2 = s_1 \circ \pi \circ s_2$ and $T_1(a_0) = a_0 + k$ $T_1(a_2) = a_2$.

$T_0 = s_2 \circ s_1 \circ \pi$ $T_1(a_1) = a_1 - k$

The extended affine Weyl (or Coxeter) group is $\widetilde{W}(A_2^{(1)}) = \langle s_0, s_1, s_2, \pi \rangle$ where

$s_j^2 = 1$, $(s_j s_{j+1})^3 = 1$, $\pi s_j = s_{j+1} \pi$ where $j \in \mathbb{N} \bmod 3$

and $\pi^3 = 1$.

$\pi$ is called a diagram automorphism (Noumi, 2004).

But behind this facade, we can interpret the roots as divisors in a space of equivalence classes of lines with an intersection form.

In this associated space, with coordinates $(f_0, f_1, f_2)$ say, the action of $s_j$ is given by with $f_0 + f_1 + f_2 = t$.

| | $f_0$ | $f_1$ | $f_2$ |
|---|---|---|---|
| $s_0$ | $f_0$ | $f_1 + \frac{a_0}{f_0}$ | $f_2 - \frac{a_0}{f_0}$ |
| $s_1$ | $f_0 - \frac{a_1}{f_1}$ | $f_1$ | $f_2 + \frac{a_1}{f_1}$ |
| $s_2$ | $f_0 + \frac{a_2}{f_2}$ | $f_1 - \frac{a_2}{f_2}$ | $f_2$ |

Letting $u_n = T_1^n(f_1)$, $v_n = T_1^n(f_0)$

we find

$$\begin{cases} u_n + u_{n+1} = t - v_n - \dfrac{a_0 + kn}{v_n} \\ v_n + v_{n-1} = t - u_n + \dfrac{a_1 - kn}{u_n} \end{cases}$$

which is the first discrete Painlevé equation or $dP_1$. (Fokas, Its & Kitaev 1992)

The case $a_0 = 0$ $k = -2$, $a_1 = -1$ was discovered by Shohat (1939) in his extension of classical orthogonal polynomials. It was rediscovered in 1980 by Bessis et al in the study of combinatorics of closed graphs, and in 1991 by Douglas et al in quantum gravity (Hermitian RMT). It is now referred to as the string equation.

We know almost nothing about its generic (highly transcendental) solutions, except for $n \to \infty$.

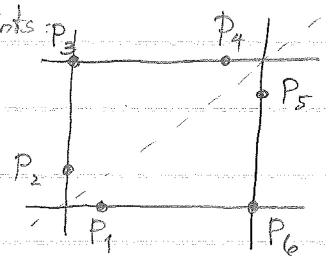
Many discrete Painlevé eqns are now known.
I will give a sketch of them:

- as integrable systems — Integrability
- how related to complex algebraic surfaces — Geometry
- special features of solutions. — Asymptotics

$L_3$ updated: §3 Geometry of Initial Value Spaces

Recall the pencil of curves (for given $a, b$)

$$f(x,y) = x^2y^2 + bxy(x+y) + ab(x+y) + a - \underset{\uparrow \text{free parameter}}{k}xy = 0$$

has base points  in $\mathbb{P}^1 \times \mathbb{P}^1$



There is a reflection symmetry in the pencil: $f(x,y) = f(y,x)$, under which

$$\{P_2, P_3, P_4\} \Leftrightarrow \{P_1, P_6, P_5\}$$

So we'll restrict our attention on blowing up $P_1, P_6, P_5$.

To blow up a point $(x,y) = (\alpha, \beta)$, means taking new coords

$$\begin{cases} x_1 = \dfrac{x-\alpha}{y-\beta} \\ y_1 = y-\beta \end{cases} \quad \text{or} \quad \begin{cases} x_2 = x-\alpha \\ y_2 = \dfrac{y-\beta}{x-\alpha} \end{cases}$$

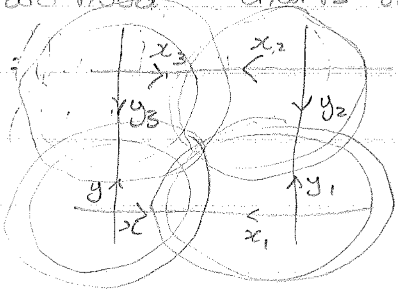
[This is the simplest transformation. In general, we can take any transf$^{n}$ that is locally analytic and invertible s.t. a new variable takes form $0/0$ at the point.]

Note: Embedding $(x_1, \dots, x_n) \in \mathbb{C}^n$, in $\mathbb{P}^n$ means that we take

$$\underbrace{[x_1, \dots, x_n, 1]}_{\text{affine coordinates}} = \left[\frac{u_1}{u_{n+1}}, \dots, \frac{u_n}{u_{n+1}}, 1\right] = \underbrace{[u_1, \dots, u_n, u_{n+1}]}_{\text{homogeneous coords}}$$

with at least one $u_j \neq 1$.

In affine coords, we need $(n+1)$ charts to cover all domains in $\mathbb{P}^n$.

We took $\mathbb{P}^1 \times \mathbb{P}^1$: 

Blowing-up $P_1$:

$$\begin{cases} x_{11} = \dfrac{x+1/6}{y} \\ y_{11} = y \end{cases} \Rightarrow \begin{matrix} x = -\frac{1}{6} + x_{11}y_{11} \\ y = y_{11} \end{matrix} \quad \text{or} \quad \begin{cases} x_{12} = x + 1/6 \\ y_{12} = \dfrac{y}{x+1/6} \end{cases} \Rightarrow \begin{matrix} x = -\frac{1}{6} + x_{12} \\ y = x_{12}y_{12} \end{matrix}$$

$$\Rightarrow f = \left(-\tfrac{1}{6} + x_{11}y_{11}\right)^2 y_{11}^2 + b\left(-\tfrac{1}{6} + x_{11}y_{11}\right)y_{11}\left(-\tfrac{1}{6} + x_{11}y_{11} + y_{11}\right) + ab\left(\cancel{-\tfrac{1}{6}} + x_{11}y_{11} + y_{11}\right) + \cancel{a}$$
$$- k\left(-\tfrac{1}{6} + x_{11}y_{11}\right)y_{11}$$

$$= y_{11}\left\{ y_{11}\left(-\tfrac{1}{6} + x_{11}y_{11}\right)^2 + b\left(-\tfrac{1}{6} + x_{11}y_{11}\right)\left(-\tfrac{1}{6} + y_{11} + x_{11}y_{11}\right) + ab(1 + x_{11}) - k\left(-\tfrac{1}{6} + x_{11}y_{11}\right) \right\}$$

No new base pts. on line $y_{11} = 0$. (Taking $-\frac{1}{6} + x_{11}y_{11} = 0 \Leftrightarrow x = 0 \to$ blowing back down to $P_1$.)

Exercise: Write $f$ in $(x_{12}, y_{12})$ to show $\nexists$ no new base pt in that chart either.

We have an exceptional line $E_1 : \{y_{11} = 0\} \cup \{x_{12} = 0\}$.

Exercise: Show that no new base pts arise from blowing up $P_5$.

Blowing-up $P_6$:

$$\begin{cases} x_{61} = \dfrac{x_1}{y_1} \\ y_{61} = y_1 \end{cases} \Rightarrow \begin{matrix} x_1 = x_{61}y_{61} \\ y_1 = y_{61} \end{matrix} \quad \text{or} \quad \begin{cases} x_{62} = x_1 \\ y_{62} = \dfrac{y_1}{x_1} \end{cases} \Rightarrow \begin{matrix} x_1 = x_{62} \\ y_1 = x_{62}y_{62} \end{matrix}$$

$$f_1 = y_1^2 + by_1(1 + x_1y_1) + ab(x_1 + x_1^2y_1) + ax_1^2 - kx_1y_1$$
$$= y_{61}^2 + by_{61}(1 + x_{61}y_{61}^2) + ab(x_{61}y_{61} + x_{61}^2y_{61}^3) + ax_{61}^2y_{61}^2 - kx_{61}y_{61}^2$$
$$= y_{61}\left\{ y_{61} + b(1 + x_{61}y_{61}^2) + ab\,x_{61}(1 + x_{61}y_{61}^2) + ax_{61}^2y_{61} - kx_{61}y_{61} \right\}$$
$$=: y_{61} f_1^{(1)}$$

$\Rightarrow$ New base pt on exceptional line $E_6 : \{y_{61} = 0\}$ given by $b + ab\,x_{61} = 0 \Rightarrow P_8 : (x_{61}, y_{61}) = \left(-\frac{1}{a}, 0\right)$

Exercise: Carry out the blow-up in chart 62 and show $P_8$ appears there on $E_6 : \{x_{62} = 0\}$ as $(x_{62}, y_{62}) = (0, -a)$.

Here $f_1^{(1)} = x_{62}y_{62}^2 + b\,y_{62}(1 + x_{62}^2y_{62}) + ab(1 + x_{62}^2y_{62}) + a\,x_{62}$
$- k\,x_{62}y_{62}$.

Blowing-up $P_8$:

$$\begin{cases} x_{81} = \dfrac{x_{62}}{y_{62}+a} \\ y_{81} = y_{62}+a \end{cases} \Rightarrow \begin{array}{l} x_{62} = x_{81}y_{81} \\ y_{62} = -a+y_{81} \end{array} \qquad \text{or} \qquad \begin{cases} x_{82} = x_{62} \\ y_{82} = \dfrac{y_{62}+a}{x_{62}} \end{cases} \Rightarrow \begin{array}{l} x_{62} = x_{82} \\ y_{62} = -a+x_{82}y_{82} \end{array}$$

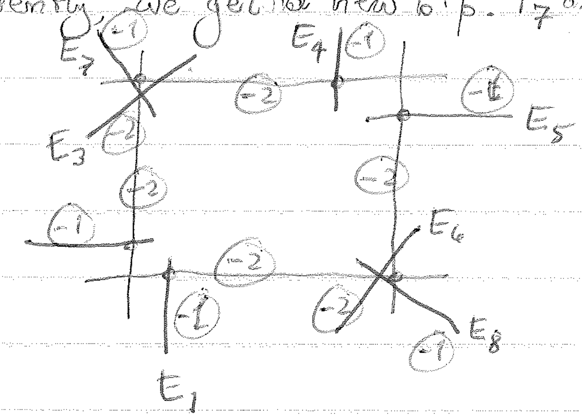
Now

$$f_1^{(1)} = y_{81}\Big\{x_{81}(-a+y_{81})^2 + b + b x_{81}^2 y_{81}(-a+y_{81})^2 + ab\,x_{81}^2 y_{81}(-a+y_{81}) + a x_{81} - k\, x_{81}(-a+y_{81})\Big\}$$

No new basepts (because coefft of $k$ doesn't vanish on $E_8 = \{y_{81}=0\}$.)

Equivalently, we get a new b.p. $P_7$ after blowing up $P_3$.

Call this resolved space $X$ $\longrightarrow$




Let $H_1 = \{x = \text{const}\} \cup \{x_1 = \text{const}\}$

$H_2 = \{y = \text{const}\} \cup \{y_2 = \text{const}\}$

The curves of self-intersection $(-2)$ are

$D_0 = E_6 - E_8$

$D_1 = H_2 - E_1 - E_6$

$D_2 = H_1 - E_2 - E_3$

$D_3 = E_3 - E_7$

$D_4 = H_2 - E_3 - E_4$

$D_5 = H_1 - E_5 - E_6$

We use the bilinear intersection form $(\ |\ )$ where

$(E_i | E_j) = 0$ if $i \neq j$ $\qquad (H_j | E_i) = 0 \quad \forall i, j$

$(E_i | E_i) = -1 \qquad (H_j | H_j) = 0$

$(H_1 | H_2) = 1$

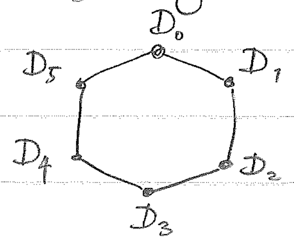
Note: $(D_0|D_1) = (E_6 - E_8 \,|\, H_2 - E_1 - E_6)$

$$= \underset{\overset{\shortparallel}{0}}{(E_6|H_2)} - \underset{\overset{\shortparallel}{0}}{(E_6|E_1)} - \underset{\overset{\shortparallel}{-1}}{(E_6|E_6)} - \underset{\overset{\shortparallel}{0}}{(E_8|H_2)} + \underset{\overset{\shortparallel}{0}}{(E_8|E_1)} + \underset{\overset{\shortparallel}{0}}{(E_8|E_6)}$$

$$= 1$$

Exercise: Calculate $(D_i|D_j)$ for $i,j = 0,\dots,5$.

Connecting $D_i$ and $D_j$ only if $(D_i|D_j) \neq 0$, we get




i.e. Dynkin diagram of type $A_5^{(1)}$.
with $D_j$ considered as simple roots.
$\delta = D_0 + D_1 + \dots + D_5$ is called the "null-root".

In the next lecture, I will show how to deduce a discrete Painlevé equation from the symmetries of this initial-value space.
For this, we need to deduce the symmetry group.

To understand this, consider the equivalence classes of all lines in $X$: $\langle H_1, H_2, E_1, \dots, E_8 \rangle = \mathrm{Pic}(X)$.
$\mathrm{Pic}(X)$ has 10-dims, while $\langle D_0, \dots, D_5 \rangle$ is only 6-dim'l.
The orthogonal subspace of $\mathrm{Pic}(X)$, orthog'l to $\langle D_0, \dots, D_5 \rangle$, is the symmetry group

So we look for $F = \mu_1 H_1 + \mu_2 H_2 + \nu_1 E_1 + \dots + \nu_8 E_8$
s.t. $(F|D_j) = 0 \quad \forall j \in \{0, \dots, 5\}$.

First generalize slightly to $P_1, \dots, P_8$ given by

(Same as above if $a_0 = b$, $a_1 = 1/b$.)

$P_1 = (-a_0^{-1}, 0)$ $\qquad$ $P_5 = (\infty, -a_1^{-1})$

$P_2 = (0, -a_1)$ $\qquad$ $P_6 = (\infty, 0)$

$P_3 = (0, \infty)$ $\qquad$ $P_7 = (0, \infty)$ s.t. $xy = -c^2 a_0 a_1 a_2^2$

$P_4 = (-a_0, \infty)$ $\qquad$ $P_8 = (\infty, 0)$ s.t. $xy = -c^2 a_0 a_1$

The story so far:

- Given $f(x,y)=0$, we resolved the base pts $P_i$, $i=1,\dots,8$, $\mapsto E_i$, $i=1,\dots,8$.
- Curves of self-intersection $-2$ were identified $\Rightarrow D_j$, $j=0,\dots,5$.
- Defined intersection form $(\,|\,)$
- Identified $\{D_j\}$ as simple roots of $A_5^{(1)}$
- Defined $\mathrm{Pic}(X) = \langle H_1, H_2, E_1, \dots, E_8\rangle$ where $H_1=\{y=\text{const}\}$, $H_2=\{x=\text{const}\}$.

In this lecture: (J., Nakazono & Shi, Reflection groups & discrete integrable systems, J. Integrable Systems (2016) 00 1-37)

- We identify the symmetry group
- Define reflections
- Deduce actions of reflections on parameters and $x,y$.
- Construct discrete Painlevé eqn.

---

Conclusion:

$L_0$: History (Shohat);
$P_{IV} \xrightarrow{BTs} dP_2$
BTs $\Leftrightarrow$ reflection groups
Outline of Sakai's description

$L_1$: Initial value spaces
Curves (from Hamiltonians)
Resolution of singularities

$L_2$: Reflection groups
Translations on affine Weyl lattices
Actions on coordinates in lattice
Birational actions on associated space

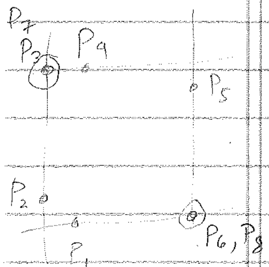
$L_3$: Geometry of IVS
Resolution of base pts
Identifying resolved space with an affine Root system.
Intersection form.

$L_4$: Constructing discrete Painlevé eqns

Q: How does this theory carry over to orthogonal polynomials?
Are OPs related to affine Weyl gps?
Is there a classification like Sakai's for all possible OPs?

# L4

Consider a slightly more general setting, space $X$ with the same group. (gives prev case if $a_0 = b$, $a_1 = 1/b$)
With $P_1, \ldots, P_8$ given by




$$\left.\begin{aligned}
P_1 &= (-a_0^{-1}, 0)\\
P_2 &= (0, -a_1)\\
P_3 &= (0, \infty)\\
P_4 &= (-a_0, \infty)\\
P_5 &= (\infty, -a_1^{-1})\\
P_6 &= (\infty, 0)\\
P_7 &= (0, \infty) \quad \text{s.t. } xy = -c^2 a_0 a_1 a_2^2\\
P_8 &= (\infty, 0) \quad \text{s.t. } xy = -c^2 a_0 a_1
\end{aligned}\right\} \Rightarrow$$

$D_0, \ldots, D_5$ forming an $A_5^{(1)}$ root-system with "null root"
$\delta = D_0 + D_1 + \ldots + D_5$
$= 2H_1 + 2H_2 - E_1 - E_2 - E_3 - \ldots - E_8$

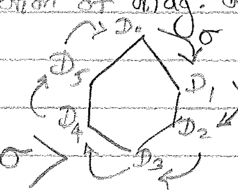
$$\begin{aligned}
D_0 &= E_6 - E_8\\
D_1 &= H_2 - E_1 - E_6\\
D_2 &= H_1 - E_2 - E_3\\
D_3 &= E_3 - E_7\\
D_4 &= H_2 - E_2 - E_4\\
D_5 &= H_1 - E_5 - E_6
\end{aligned}$$

In the Picard group generated by equivalence classes of lines $H_1, H_2, E_1, \ldots, E_8$ this group is orthogonal to two other orthogonal subgroups

$$\begin{cases}
\alpha_0 = H_1 - E_1 - E_4\\
\alpha_1 = H_2 - E_2 - E_5\\
\alpha_2 = H_1 + H_2 - E_3 - E_6 - E_7 - E_8
\end{cases}
\quad \text{and} \quad
\begin{cases}
\beta_0 = H_1 + H_2 - E_2 - E_4 - E_6 - E_8\\
\beta_1 = H_1 + H_2 - E_1 - E_3 - E_5 - E_7
\end{cases}$$

$\Downarrow$ $A_2^{(1)}$ Check: e.g. $(D_j | \alpha_0) = 0$
$j=1 \Rightarrow (H_1|H_2) + (E_1|E_1) = 1 - 1 = 0$ $\Downarrow$ $A_1^{(1)}$.

So the initial value space $X$ is unchanged under the action of $(A_2 + A_1)^{(1)}$ which forms a symmetry group. Also under the action of diag. automorphism




Consider the extended affine Weyl group

$$\widetilde{W}\left((A_2 + A_1)^{(1)}\right) = \langle s_0, s_1, s_2, w_0, w_1, \sigma \rangle$$

Here the action of $s_i$ is reflection by $\alpha_i$
$w_j$ " " by $\beta_j$.

Convention
$f.w = w^{-1}.f = \bar{f}$

Each reflection is given by the intersection form: take any elt $v$ in $Pic(X)$

$$v.s_i = v - 2\frac{(v|\alpha_i)}{(\alpha_i|\alpha_i)}\alpha_i$$

$$v.w_j = v - 2\frac{(v|\beta_j)}{(\beta_j|\beta_j)}\beta_j$$

We can work out the effects of these actions on the parameters $\{a_0, a_1, a_2, c\}$ and the coordinates $x, y$.

E.g. ~~take $s_0(x) = \bar{x}$, $s_0(y) = \bar{y}$, $s(a_i) = \bar{a}_i$, $\bar{c} = s_0(c)$.~~
Then noting

$$(\alpha_0|\alpha_0) = (H_1 - E_1 - E_4 | H_1 - E_1 - E_4)$$
$$= -1-1 = -2,$$

and

$$(H_1|\alpha_0) = 0, \quad (H_2|\alpha_0) = 1, \quad (E_j|\alpha_0) = 0 \text{ for } j \neq 1,4.$$

$H_2 . s_0 = H_2 - 2\frac{(H_2|\alpha_0)}{(\alpha_0|\alpha_0)}\alpha_0$
$= H_2 - \frac{2}{(-2)}(H_2|H_1 - E_1 - E_4)(H_1 - E_4 - E_4)$
$= H_2 + H_1 - E_4 - E_1$

we get

$$H_1 . s_0 = \overline{H}_1 \quad \text{and} \quad E_j . s_0 = \overline{E}_j$$
$$H_2 . s_0 = \overline{H}_2 + \overline{H}_1 - \overline{E}_1 - \overline{E}_4$$

← notation for image of action

which means the lines $\{y = \text{const}\}$ are mapped birationally to $\{\bar{y} = \text{const}\}$

$mH_1 + nH_2 - \sum_{i=1}^{8} \mu_i E_i$ is a curve of bi-deg $(m,n)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ passing through base pts $P_i$ with mult. $\mu_i$.

i.e.

$$\bar{x}(A_1 x + A_2) + A_3 x + A_4 = 0$$

and similarly the $\overset{\deg(0,1)}{\text{lines}}$ $\{x = \text{const}\}$ are mapped to $\overset{\deg(1,1)}{\text{lines}}$ through $P_1$ and $P_4$.

i.e.

$$\bar{y}\big(\underbrace{B_1 y(x + a_0)}_{\text{from } P_4} + \underbrace{B_2(x + a_0^{-1})}_{\text{from } P_1}\big) + \underbrace{B_3 y(x + a_0)}_{\text{from } P_4} + \underbrace{B_4(x + a_0^{-1})}_{\text{from } P_1} = 0$$

$E_2$: (from $P_2 = (0, -a_1)$) stays unchanged.

$$\Rightarrow \quad x = 0 \quad \text{maps to} \quad \bar{x} = 0 \quad \Rightarrow \quad A_4 = 0$$

from $E_5$: $x = \infty$ " " $\bar{x} = \infty \Rightarrow A_1 = 0$

and so on for $E_2, E_3, E_5$ and $E_6$:

$$\Rightarrow \text{ we find } B_1 = B_4 = 0 \text{ while } B_2 = -a_0 B_3, \quad \boxed{\bar{a}_1 = a_0 a_1.}$$

from

$P_2 = (0, -a_1)$:
$\bar{y}(B_1 \cdot -a_1 a_0 + B_2 \cdot a_0^{-1}) + B_3 \cdot (-a_1) a_0 + B_4 \cdot a_0^{-1} = 0$

$P_3 : (0, \infty)$
$\bar{y}(B_1 a_0) + B_3 a_0 = 0$

and from $E_7, E_8$:

$$\bar{x} = -\frac{A_3}{A_2}x, \quad \bar{y} = \frac{y(x + a_0)}{a_0(x + a_0^{-1})} = \frac{y(a_0 + x)}{(1 + a_0 x)}$$

$$\Rightarrow \quad \bar{x}\,\bar{y} = \frac{-A_3}{A_2}\,\frac{(a_0+x)}{(1+a_0x)}\,xy$$

$$\sim \frac{-A_3}{A_2}\cdot\frac{-c^2 a_0 a_1}{a_0} \quad \text{at } P_8$$

$\Rightarrow$

$$-\bar{c}^2\,\bar{a}_0\,\bar{a}_1 = +c^2\,a_1\,\frac{A_3}{A_2}$$

i.e.
$$\frac{A_3}{A_2} = -\frac{\bar{c}^2\bar{a}_0 a_0}{c^2}$$

while at $P_7$, we get

$$-\bar{c}^2\,\bar{a}_0\bar{a}_1\,\bar{a}_2^2 = \frac{-A_3}{A_2}a_0\cdot -c^2 a_0 a_1 a_2^2$$

$$\Rightarrow \frac{A_3}{A_2} = -\frac{\bar{c}^2}{c^2}\,\frac{\bar{a}_0 a_0 a_1\,\bar{a}_2^2}{a_0^2 a_1 a_2^2} = -\frac{\bar{c}^2}{c^2}\bar{a}_0 a_0$$

$$\Rightarrow \bar{a}_2^2 = a_0^2 a_2^2 .$$ We take $\boxed{\bar{a}_2 = a_0 a_2}$.

Now consider

$E_1\cdot s_0 = \bar{E}_1 + \bar{H}_1 - \bar{E}_1 - \bar{E}_4 = \bar{H}_1 - \bar{E}_4 \quad \Rightarrow P_1$ is mapped to $P_4$

$E_4\cdot s_0 = \bar{H}_1 - \bar{E}_1 \quad$ (check!) $\quad \Rightarrow P_4$ is mapped to $P_1$

$$-\bar{a}_0 = \frac{-A_3}{A_2}\cdot\frac{-1}{a_0} \quad \text{and} \quad \frac{-1}{\bar{a}_0} = \frac{-A_3}{A_2}\cdot -a_0$$

$$\Rightarrow \bar{a}_0^2 = \frac{1}{a_0^2} \Rightarrow \boxed{\bar{a}_0 = \frac{1}{a_0}} \Rightarrow \frac{A_3}{A_2} = -1 .$$

and

$$-\bar{a}_0 = \frac{\bar{c}^2}{c^2}\bar{a}_0 a_0 \times \frac{-1}{a_0} = -\frac{\bar{c}^2}{c^2}\bar{a}_0 \Rightarrow \bar{c} = c .$$

So far we have shown

$$s_0 : (a_0, a_1, a_2, c) \longmapsto \left(\frac{1}{a_0}, a_0 a_1, a_0 a_2, c\right)$$

and $\quad s_0(x) = x \quad , \quad s_0(y) = \dfrac{y(a_0+x)}{1+a_0x}$.

Note: $\quad \bar{a}_0\bar{a}_1\bar{a}_2 = \frac{1}{a_0}\cdot a_0 a_1 \cdot a_0 a_2 = a_0 a_1 a_2 =: q \quad$ const.

These are the building blocks that lead to a discrete Painlevé equation. Taking $\pi = \sigma^2$, $r = \sigma^3$ and defining translations on $\widetilde{W}((A_2+A_1)^{(1)})$ by

$T_1 = \pi\, s_2\, s_1$
$T_2 = \pi\, s_0\, s_2$
$T_3 = \pi\, s_1\, s_0$
$T_4 = r\, w_0$

we get under repeated composition of $T_1$: $F_n = T_1^n(x)$, $G_n = T_1^n(y)$

$$\begin{cases} G_{n+1}\, G_n = \dfrac{q c^2}{F_n} \dfrac{(1 + q^n a_0 F_n)}{(q^n a_0 + F_n)} \\[2ex] F_{n+1}\, F_n = \dfrac{q c^2}{G_{n+1}} \dfrac{(1 + q^n a_0 a_2 G_{n+1})}{(q^n a_0 a_2 + G_{n+1})} \end{cases}$$

which is a q-discrete Painlevé equation.

(See JNS 2016 J. Integ. Systems)

Similar for $T_2$, $T_3$.
But $T_4$ leads to $q$-$P_{IV}$
and half-translation $R_1 = \pi^2 s_1$, where $R_1^2 = T_1$, leads to $q P_{II}$.