Multiple Orthogonal Polynomials

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Plan of the course

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lecture 5: Riemann-Hilbert problem

Hermite-Padé approximation

Let $(f_1, \ldots, f_r)$ be r Markov functions, i.e.,

$$f_j(z) = \int \frac{d\mu_j(x)}{z - x} = \sum_{k=0}^{\infty} \frac{m_k^{(j)}}{z^{k+1}}.$$

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Definition (Type I Hermite-Padé)

Type I Hermite-Padé approximation is to find r polynomials $(A_{\vec{n},1}, \ldots, A_{\vec{n},r})$, with $\deg A_{\vec{n},j} \leq n_j - 1$, and a polynomial $B_{\vec{n}}$ such that

$$\sum_{j=1}^{r} A_{\vec{n},j}(z) f_j(z) - B_{\vec{n}}(z) = \mathcal{O}\left(\frac{1}{z^{|\vec{n}|}}\right), \qquad z \to \infty.$$

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$$B_{\vec{n}}(z) = \int \sum_{j=1}^{r} \frac{A_{\vec{n},j}(z) - A_{\vec{n},j}(x)}{z-x} \, d\mu_j(x).$$

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Definition (Type II Hermite-Padé)

Type II Hermite-Padé approximation is to find a polynomial $P_{\vec{n}}$ of degree $\leq |\vec{n}|$ and polynomials $Q_{\vec{n},1}, \ldots, Q_{\vec{n},r}$ such that

$$P_{\vec{n}}(z) f_j(z) - Q_{\vec{n},j}(z) = \mathcal{O}\left(\frac{1}{z^{n_j+1}}\right), \qquad z \to \infty,$$

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$$Q_{\vec{n},j}(z) = \int \frac{P_{\vec{n}}(z) - P_{\vec{n}}(x)}{z-x}\, d\mu_j(x).$$

Multiple Hermite polynomials

The type II multiple Hermite polynomials $H_{\vec{n}}$ satisfy

$$\int_{-\infty}^{\infty} H_{\vec{n}}(x) x^k e^{-x^2+c_j x}\, dx = 0, \qquad 0 \leq k \leq n_j - 1$$

for $1 \leq j \leq r$, with $c_i \neq c_j$ whenever $i \neq j$.

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Rodrigues formula:

$$e^{-x^2} H_{\vec{n}}(x) = \frac{(-1)^{|\vec{n}|}}{2^{|\vec{n}|}} \left(\prod_{j=1}^{r} e^{-c_j x} \frac{d^{n_j}}{dx^{n_j}} e^{c_j x} \right) e^{-x^2}.$$

Multiple Hermite polynomials

Explicit expression:

$$H_{\vec{n}}(x) = \frac{(-1)^{|\vec{n}|}}{2^{|\vec{n}|}} \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \binom{n_1}{k_1} \cdots \binom{n_r}{k_r} c_1^{n_1-k_1} \cdots c_r^{n_r-k_r} (-1)^{|\vec{k}|} H_{|\vec{k}|}(x),$$

where H_n are the usual Hermite polynomials.

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Nearest neighbor recurrence relations:

$$xH_{\vec{n}}(x) = H_{\vec{n}+\vec{e}_k}(x) + \frac{c_k}{2} H_{\vec{n}}(x) + \frac{1}{2} \sum_{j=1}^{r} n_j H_{\vec{n}-\vec{e}_j}(x), \qquad 1 \leq k \leq r.$$

Differential properties

Raising operators:

$$\left(e^{-x^2+c_jx} H_{\vec{n}-\vec{e}_j}(x)\right)' = -2e^{-x^2+c_jx} H_{\vec{n}}(x), \qquad 1 \le j \le r.$$

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Lowering operator:

$$H'_{\vec{n}}(x) = \sum_{j=1}^{r} n_j H_{\vec{n}-\vec{e}_j}(x).$$

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Differential equation:

$$\left(\prod_{j=1}^{r} D_j\right) DH_{\vec{n}}(x) = -2\left(\sum_{j=1}^{r} n_j \prod_{i\neq j} D_i\right) H_{\vec{n}}(x),$$

where

$$D = \frac{d}{dx}, \qquad D_j = e^{x^2-c_jx}De^{-x^2+c_jx}$$

Integral representations

Type II multiple Hermite:

$$H_{\vec{n}}(x) = \frac{1}{\sqrt{\pi} i} \int_{-i\infty}^{i\infty} e^{(s-x)^2} \prod_{j=1}^{r} \left(s - \frac{c_j}{2} \right)^{n_j} \, ds.$$

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Type I multiple Hermite:

$$e^{-x^2+c_k x} A_{\vec{n},k}(x) = \frac{1}{\sqrt{\pi} 2\pi i} \oint_{\Gamma_k} e^{-(t-x)^2} \prod_{j=1}^{r} \left(t - \frac{c_j}{2}\right)^{-n_j} dt$$

where Γ_k is a closed contour encircling $c_k/2$ once and none of the other $c_j/2$.

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where Γ_k is a closed contour encircling $c_k/2$ once and none of the other $c_j/2$.

$$Q_{\vec{n}}(x) = \sum_{k=1}^{r} e^{-x^2 + c_k x} A_{\vec{n},k}(x) = \frac{1}{\sqrt{\pi} 2\pi i} \oint_{\Gamma} e^{-(t-x)^2} \prod_{j=1}^{r} \left(t - \frac{c_j}{2} \right)^{-n_j} \, dt$$

where Γ is a closed contour encircling all $c_j/2$.

Random matrices with external source

Let M be a random Hermitian matrix of size $N \times N$, and consider the ensemble with probability distribution

$$\frac{1}{Z_N} \exp\Big(-\mathrm{Tr}(M^2 - AM)\Big)\, dM, \qquad dM = \prod_{i=1}^{N} dM_{i,i} \prod_{1 \le i < j \le N} dM_{i,j}$$

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Property

Suppose A has eigenvalues $c_1, \ldots, c_r$ with multiplicities $n_1, \ldots, n_r$, then

$$\mathbb{E}\Big(\det(M - zI_N)\Big) = (-1)^{|\vec{n}|} H_{\vec{n}}(z).$$

Random matrices with external source

Property

The density of the eigenvalues is given by

$$P_N(\lambda_1, \ldots, \lambda_N) = \frac{1}{N!} \det\Big(K_N(\lambda_i, \lambda_j)\Big)_{i,j=1}^{N},$$

where the kernel is given by

$$K_N(x,y) = e^{-(x^2+y^2)/2} \sum_{k=0}^{N-1} H_{\vec{n}_k}(x) Q_{\vec{n}_{k+1}(y)},$$

with $(\vec{n}_k)_{0 \leq k \leq N}$ *a path from* $\vec{0}$ *to* $\vec{n}$ *in* $\mathbb{N}^r$ *and*

$$Q_{\vec{n}}(y) = \sum_{j=1}^{r} A_{\vec{n},j}(y) e^{c_j y}.$$

Random matrices with external source

Property

The m-point correlation function

$$R_m(\lambda_1, \ldots, \lambda_m) = \frac{N!}{(N-m)!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P_N(\lambda_1, \ldots, \lambda_N)\, d\lambda_{m+1} \ldots d\lambda_N$$

is given by

$$R_m(\lambda_1, \ldots, \lambda_m) = \det\Big(K_N(\lambda_i, \lambda_j) \Big)_{i,j=1}^{m},$$

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Non-intersecting Brownian motions

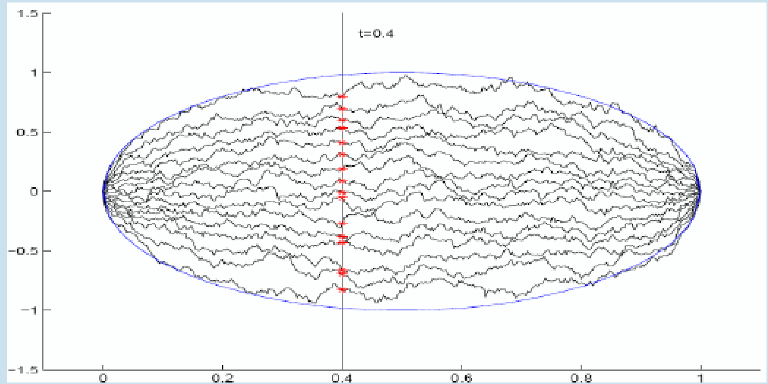

Figure: Non-intersecting Brownian motions

Non-intersecting Brownian motions

Density of the probability that the n non-intersecting paths, leaving $(t=0)$ at $a_1,\ldots,a_n$ and arriving $(t=1)$ at $b_1,\ldots,b_n$ are at $x_1,\ldots,x_n$ at time $t \in (0,1)$ is[1]

$$p_{n,t}(x_1,\ldots,x_n) = \frac{1}{Z_n} \det\Big(P(t,a_j,x_k)\Big)_{j,k=1}^{n} \det\Big(P(1-t,b_j,x_k)\Big)_{j,k=1}^{n}$$

$$P(t,a,x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}(x-a)^2}.$$

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$$P(t,a,x)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{1}{2t}(x-a)^2}.$$

When $a_1,\ldots,a_n\to 0$ and $b_1,\ldots,b_n\to 0$ then

$$p_{n,t}(x_1,\ldots,x_n)=\frac{1}{n!}\det\Big(K_n(x_j,x_k)\Big)_{j,k=1}^{n}$$

$$K_n(x,y)=e^{-\frac{x^2}{4t}-\frac{y^2}{4(1-t)}}\sum_{k=0}^{n-1}H_k\Big(\frac{x}{\sqrt{2t}}\Big)H_k\Big(\frac{y}{\sqrt{2(1-t)}}\Big)$$

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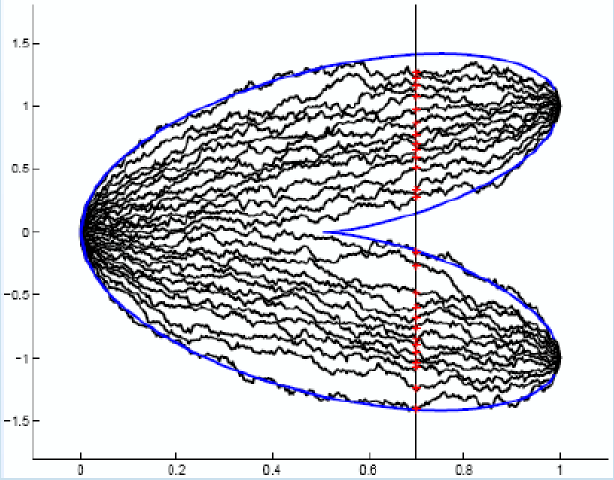

Figure: Non-intersecting Brownian motions (two arriving points)

When $a_1, \ldots, a_n \to 0$ and $b_1, \ldots, b_{n/2} \to -b$, $b_{n/2+1}, \ldots, b_n \to b$ then

$$p_{n,t}(x_1, \ldots, x_n) = \frac{1}{n!} \det\Big(K_n(x_j, x_k)\Big)_{j,k=1}^{n}$$

$$K_n(x,y) = e^{-\frac{x^2}{4t} - \frac{y^2}{4(1-t)}} \sum_{k=0}^{n-1} H_{\vec{n}_k}\Big(\frac{x}{\sqrt{2t}}\Big) Q_{\vec{n}_{k+1}}\Big(\frac{y}{\sqrt{2(1-t)}}\Big)$$

with multiple orthogonal polynomials for the weights

$$e^{-x^2-2bx}, \quad e^{-x^2+2bx}$$

Non-intersecting squared Bessel paths

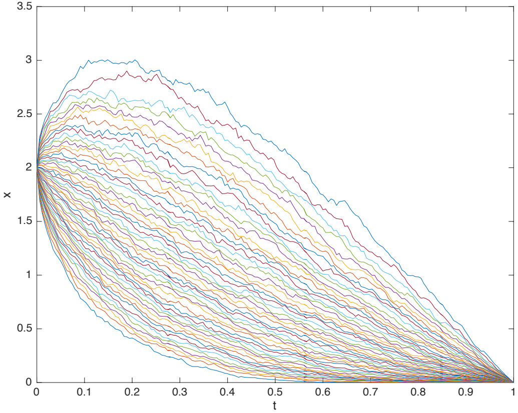

Figure: Non-intersecting squared Bessel paths

References

P.M. Bleher, A.B.J. Kuijlaars, **Integral representations for multiple Hermite and multiple Laguerre polynomials**, Ann. Inst. Fourier, Grenoble **55** (2005), no. 6, 2001–2014.

P.M. Bleher, A.B.J. Kuijlaars, **Random matrices with external source and multiple orthogonal polynomials**, International Mathematics Research Notices **2004**, no. 3, 109–129.

E. Daems, A.B.J. Kuijlaars, **Multiple orthogonal polynomials of mixed type and non-intersecting Brownian motions**, J. Approx. Theory **146** (2007), no. 1, 91–114.

M.E.H. Ismail, **Classical and Quantum Orthogonal Polynomials in One Variable**, Encyclopedia of Mathematics and its Applications 98, Cambridge University Press, 2005 (paperback edition 2009) ; in particular **Chapter 23**.

W. Van Assche, J.S. Geronimo, A.B.J. Kuijlaars, **Riemann-Hilbert problems for multiple orthogonal polynomials**, in 'Special Functions 2000: Current Perspective and Future Directions', Kluwer, Dordrecht, 2001, pp. 23–59.