Multiple Orthogonal Polynomials

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Plan of the course

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lecture 1: Definitions + basic properties

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Jacobi-Angelesco + Jacobi-Piñeiro

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lecture 5: Riemann-Hilbert problem

Multiple Laguerre polynomials

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$$w(x) = x^\alpha e^{-x}, \qquad x \in [0, \infty), \ \alpha > -1.$$

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   This gives **multiple Laguerre polynomials of the first kind**

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There are two easy ways to obtain multiple Laguerre polynomials:

1. Changing the parameter α to $\alpha_1, \ldots, \alpha_r$
   This gives **multiple Laguerre polynomials of the first kind**
2. Changing the exponential decay at infinity from e^{-x} to $e^{-c_j x}$ with parameters $c_1, \ldots, c_r$
   This gives **multiple Laguerre polynomials of the second kind**

Type II multiple Laguerre of the first kind: $L_{\vec{n}}^{\vec{\alpha}}(x)$

$$\int_0^\infty x^k L_{\vec{n}}^{\vec{\alpha}}(x) x^{\alpha_j} e^{-x}\, dx = 0, \qquad 0 \leq k \leq n_j - 1,$$

for $1 \leq j \leq r$.

Multiple Laguerre I

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Parameters $\alpha_j > -1$ and $\alpha_i - \alpha_j \notin \mathbb{Z}$ whenever $i \neq j$. (AT system)

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Rodrigues formula:

$$(-1)^{|\vec{n}|} e^{-x} L_{\vec{n}}^{\vec{\alpha}}(x) = \prod_{j=1}^{r} \left(x^{-\alpha_j} \frac{d^{n_j}}{dx^{n_j}} x^{n_j+\alpha_j} \right) e^{-x}.$$

Multiple Laguerre I

Explicit formula:

$$L_{\vec{n}}^{\vec{\alpha}}(x) = \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} (-1)^{|\vec{k}|} \frac{n_1!}{(n_1-k_1)!} \cdots \frac{n_r!}{(n_r-k_r)!}$$

$$\times \binom{n_r+\alpha_r}{k_r} \binom{n_r+n_{r-1}+\alpha_{r-1}-k_r}{k_{r-1}} \cdots \binom{|\vec{n}|-|\vec{k}|+k_1+\alpha_1}{k_1} x^{|\vec{n}|-|\vec{k}|}$$

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$$(-1)^{|\vec{n}|} e^{-x} L_{\vec{n}}^{\vec{\alpha}}(x) = \prod_{j=1}^{r} (\alpha_j+1)_{n_j} \; {}_rF_r \left(\begin{array}{c} n_1+\alpha_1+1, \ldots, n_r+\alpha_r+1 \\ \alpha_1+1, \ldots, \alpha_r+1 \end{array} \middle| -x \right).$$

Multiple Laguerre I: recurrence relation

Recurrence relation:

$$xL_{\vec{n}}(x) = L_{\vec{n}+\vec{e}_k}(x) + b_{\vec{n},k} L_{\vec{n}}(x) + \sum_{j=1}^{r} a_{\vec{n},j} L_{\vec{n}-\vec{e}_j}(x)$$

$$a_{\vec{n},j} = n_j(n_j+\alpha_j) \prod_{i=1, i\neq j}^{r} \frac{n_j+\alpha_j-\alpha_i}{n_j-n_i+\alpha_j-\alpha_i},$$

$$b_{\vec{n},k} = |\vec{n}| + n_k + \alpha_k + 1.$$

Raising operators:

$$\frac{d}{dx}\left(x^{\alpha_j+1}e^{-x}L_{\vec{n}-\vec{e}_j}^{\vec{\alpha}+\vec{e}_j}(x)\right) = -x^{\alpha_j}e^{-x}L_{\vec{n}}^{\vec{\alpha}}(x), \qquad 1 \leq j \leq r.$$

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Lowering operator:

$$\frac{d}{dx}L_{\vec{n}}^{\vec{\alpha}}(x) = \sum_{j=1}^{r} \frac{\prod_{i=1}^{r}(n_i+\alpha_i-\alpha_j)}{\prod_{i=1,i\neq j}^{r}(\alpha_i-\alpha_j)} L_{\vec{n}-\vec{e}_j}^{\vec{\alpha}+\vec{e}_j}(x).$$

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Differential equation:

$$\left(\prod_{j=1}^{r}D_j\right)DL_{\vec{n}}^{\vec{\alpha}}(x) = -\sum_{j=1}^{r}\frac{\prod_{i=1}^{r}(n_i+\alpha_i-\alpha_j)}{\prod_{i=1,i\neq j}^{r}(\alpha_i-\alpha_j)}\left(\prod_{i\neq j}D_i\right)L_{\vec{n}}^{\vec{\alpha}}(x).$$

$$D = \frac{d}{dx}, \qquad D_j = x^{-\alpha_j}e^{x}Dx^{\alpha_j+1}e^{-x}.$$

Multiple Laguerre I: integral representations

Integral representations:

$$e^{-x} L_{\vec{n}}^{\vec{\alpha}}(x) = \frac{(-1)^{|\vec{n}|}}{2\pi i} \oint_{\Sigma} x^{-z-1} \Gamma(z+1) \prod_{j=1}^{r} (\alpha_j - z)_{n_j} \, dz$$

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Γ is a contour enclosing $\alpha_j, \alpha_j + 1, \ldots, \alpha_j + n_j - 1$ for $1 \leq j \leq r$.

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$$(-1)^{|\vec{n}|} e^{-x} L_{\vec{n}}^{\vec{\alpha}}(x) = \frac{n_1! \cdots n_r!}{(2\pi i)^r}$$

$$\times \oint_{\Sigma} \cdots \oint_{\Sigma} \exp\left(-\frac{x}{t_1 \ldots t_r}\right) \frac{t_1^{-\alpha_1-1} \cdots t_r^{-\alpha_r-1}}{(1-t_1)^{n_1+1} \cdots (1-t_r)^{n_r+1}} \, dt_1 \cdots dt_r$$

Multiple Laguerre I: zero distribution

Theorem (E+J Coussement, VA)

Let $0 < x_{1,2n} < x_{2,2n} < \cdots < x_{2n,2n}$ *be the zeros of* $L_{n,n}^{(\alpha_1,\alpha_2)}$. *Then*

$$\lim_{n\to\infty} \frac{1}{2n} \sum_{k=1}^{2n} f\left(\frac{x_{k,2n}}{2n}\right) = \int_0^{27/4} f(x) w_2(x)\, dx,$$

for every continuous function f *on* $[0, 27/4]$, *where* $w_2(x) = 4/27 g(4x/27)$ *with*

$$g(x) = \frac{2\sqrt{3}}{16\pi} \frac{(1+3\sqrt{1-x})(1-\sqrt{1-x})^{1/3} - (1-3\sqrt{1-x})(1+\sqrt{1-x})^{1/3}}{x^{2/3}}.$$

Multiple Laguerre I: zero distribution

The proof uses the four-term recurrence relation

$$xp_n(x) = p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x) + d_n p_{n-2}(x),$$

$$p_{2n}(x) = L_{n,n}(x), \quad p_{2n+1}(x) = L_{n+1,n}(x)$$

where

$$b_{2n} = 3n + \alpha_1 + 1, \quad b_{2n+1} = 3n + \alpha_2 + 2,$$
$$c_{2n} = n(3n + \alpha_1 + \alpha_2), \quad c_{2n+1} = 3n^2 + n(\alpha_1 + \alpha_2 + 3) + \alpha_1 + 1,$$
$$d_{2n} = n(n + \alpha_1)(n + \alpha_1 - \alpha_2), \quad d_{2n+1} = n(n + \alpha_2)(n + \alpha_2 - \alpha_1).$$

Multiple Laguerre I: zero distribution

Theorem (Neuschel, VA)

Let $0 < x_{1,rn} < x_{2,rn} < \cdots < x_{rn,rn}$ *be the zeros of* $L^{\vec{\alpha}}_{n,n,\ldots,n}$. *Then*

$$\lim_{n\to\infty} \frac{1}{rn} \sum_{k=1}^{rn} f\left(\frac{x_{k,rn}}{rn}\right) = \int_0^{c_r} f(x/r) u_r(x)\, dx, \qquad c_r = \frac{(r+1)^{r+1}}{r^r},$$

where u_r *is given by*

$$u_r(x) = \frac{1}{r\pi} \frac{(\sin r\varphi)^{r+1}}{\bigl(\sin(r+1)\varphi\bigr)^r},$$

where

$$x = \frac{\bigl(\sin(r+1)\varphi\bigr)^{r+1}}{\sin\varphi (\sin r\varphi)^r}, \qquad 0 < \varphi < \frac{\pi}{r+1}.$$

Multiple Laguerre polynomials: zero distribution

Theorem (Van Assche)

Let $\vec{n} = (\lfloor q_1 n \rfloor, \ldots, \lfloor q_r n \rfloor)$, *where* $q_j > 0$ $(1 \leq j \leq r)$ *and* $q_1 + \cdots + q_r = 1$. *Suppose*

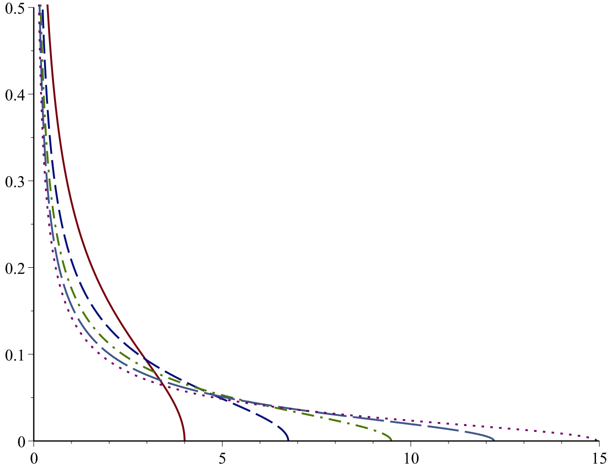
$$\lim_{n\to\infty} \frac{a_{\vec{n},j}}{n^{2\gamma}} = a_j, \quad \lim_{n\to\infty} \frac{b_{\vec{n},k}}{n^{\gamma}} = b_j.$$

Then uniformly on compact sets of $\mathbb{C} \setminus \mathbb{R}$

$$\lim_{n\to\infty} \frac{P_{\vec{n}+\vec{e}_k}(n^\gamma x)}{n^\gamma P_{\vec{n}}(n^\gamma x)} = z - b_k,$$

where z *is the solution of the algebraic equation*

$$x - z = \sum_{j=1}^{r} \frac{a_j}{z - b_j}, \qquad \lim_{x\to\infty} z - x = 0.$$



$r = 1$ (solid), $r = 2$ (dash), $r = 3$ (dash dot), $r = 4$ (long dash), $r = 5$ (point)

Multiple Laguerre II

Type II multiple Laguerre polynomials of the second kind $L_{\vec{n}}^{\alpha,\vec{c}}(x)$

$$\int_0^\infty x^k L_{\vec{n}}^{\alpha,\vec{c}}(x) x^\alpha e^{-c_j x}\, dx = 0, \qquad 0 \leq k \leq n_j - 1,$$

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Parameters $\alpha > -1$ and $c_j > 0$ with $c_i \neq c_j$ whenever $i \neq j$.

Rodrigues formula:

$$(-1)^{|\vec{n}|} \prod_{j=1}^r c_j^{n_j}\; x^\alpha L_{\vec{n}}^{\alpha,\vec{c}}(x) = \prod_{j=1}^r \left(e^{c_j x} \frac{d^{n_j}}{dx^{n_j}} e^{-c_j x} \right)\; x^{|\vec{n}|+\alpha}.$$

Multiple Laguerre II

Explicit expression:

$$L_{\vec{n}}^{\alpha,\vec{c}}(x) = \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \binom{n_1}{k_1} \cdots \binom{n_r}{k_r} \binom{|\vec{n}|+\alpha}{|\vec{k}|} (-1)^{|\vec{k}|} \frac{|\vec{k}|!}{c_1^{k_1} \cdots c_r^{k_r}} x^{|\vec{n}|-|\vec{k}|}.$$

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Recurrence relations:

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$$a_{\vec{n},j} = \frac{n_j(|\vec{n}|+\alpha)}{c_j^2}, \quad b_{\vec{n},k} = \frac{|\vec{n}|+\alpha+1}{c_k} + \sum_{j=1}^{r} \frac{n_j}{c_j}.$$

Multiple Laguerre II: differential properties

Raising operators:

$$\frac{d}{dx} x^{\alpha+1} e^{-c_j x} L_{\vec{n}-\vec{e}_j}^{\alpha+1,\vec{c}}(x) = -c_j x^{\alpha} e^{-c_j x} L_{\vec{n}}^{\alpha,\vec{c}}(x), \qquad 1 \leq j \leq r.$$

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Lowering operator:

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Differential equation:

$$\left(\prod_{j=1}^{r} D_j\right) x^{\alpha+1} D L_{\vec{n}}^{\alpha,\vec{c}}(x) = -\sum_{j=1}^{r} c_j n_j \left(\prod_{i \neq j} D_i\right) x^{\alpha} L_{\vec{n}}^{\alpha,\vec{c}}(x),$$

$$D = \frac{d}{dx}, \quad D_j = e^{c_j x} D e^{-c_j x}.$$

Mutiple Laguerre II: integral representations

Integral representation for type II: (integer α)

$$x^\alpha L_{\vec{n}}^{\alpha,\vec{c}}(x) = (-1)^{|\vec{n}|} \frac{(|\vec{n}|+\alpha)!}{c_1^{n_1}\cdots c_r^{n_r}} \frac{1}{2\pi i} \oint_\Gamma e^{xs} s^{-|\vec{n}|-\alpha-1} \prod_{j=1}^r (s-c_j)^{n_j}\, ds,$$

Γ a contour around 0 which does not encircle any of the c_j.

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Γ a contour around 0 which does not encircle any of the c_j.
Integral representation for type I: (α integer)

$$e^{-c_j x} A_{\vec{n},k}(x) = (-1)^{|\vec{n}|+1} \frac{c_1^{n_1}\cdots c_r^{n_r}}{(|\vec{n}|+\alpha-1)!} \frac{1}{2\pi i} \oint_{\Gamma_k} e^{-xt} t^{|\vec{n}|+\alpha-1} \prod_{j=1}^r (t-c_j)^{-n_j}$$

Γ_k a contour around c_k, does not enclose 0 or any other c_j $(j \neq k)$.

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$$Q_{\vec{n}}(x) = (-1)^{|\vec{n}|+1} \frac{c_1^{n_1} \cdots c_r^{n_r}}{(|\vec{n}|+\alpha-1)!} \frac{1}{2\pi i} \oint_{\Gamma} e^{-xt} t^{|\vec{n}|+\alpha-1} \prod_{j=1}^{r} (t-c_j)^{-n_j} \, dt,$$

Γ a contour encircling $c_1, \ldots, c_r$ but not 0.

Random matrices: Wishart ensemble

John Wishart (1928) introduced the **Wishart distribution** for $N \times N$ positive definite Hermitian matrices

$$M = XX^*, \qquad X \in \mathbb{C}^{N \times (N+p)}$$

where all the columns of X are independent and have a multivariate Gauss distribution with covariance matrix Σ.

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$$\frac{1}{Z_N} e^{-\mathrm{Tr}(\Sigma^{-1} M)} (\det M)^p \, dM.$$

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$$\mathbb{E}\Big(\det(M - zI_N)\Big) = (-1)^{|\vec{n}|} L_{\vec{n}}^{p,\vec{c}}(z).$$

Multiple Laguerre II: asymptotics

Riemann-Hilbert approach (Lysov and Wielonsky, 2008) for $r = 2$.

Asymptotic behavior for $P_n(z) = n^{-2n} L_{n,n}^{\alpha,(c_1,c_2)}(nz)$

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Cubic equation

$$z = \frac{2c_1c_2 - (c_1 + c_2)\psi}{\psi(\psi - c_1)(\psi - c_2)}$$

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Three solutions, which for $z \to \infty$

$$\psi_0(z) = \frac{2}{z} + \mathcal{O}(1/z^2),$$

$$\psi_1(z) = c_1 - \frac{1}{z} + \mathcal{O}(1/z^2),$$

$$\psi_2(z) = c_2 - \frac{1}{z} + \mathcal{O}(1/z^2).$$

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Three solutions, which for $z \to \infty$

$$\psi_0(z) = \frac{2}{z} + \mathcal{O}(1/z^2),$$

$$\psi_1(z) = c_1 - \frac{1}{z} + \mathcal{O}(1/z^2),$$

$$\psi_2(z) = c_2 - \frac{1}{z} + \mathcal{O}(1/z^2).$$

Zero distribution ν_1 determined by ψ_0:

$$\int \log(z - x)\, d\nu_1(x) = \int^z \psi_0(s)\, ds$$

Multiple Laguerre II: asymptotics

Let

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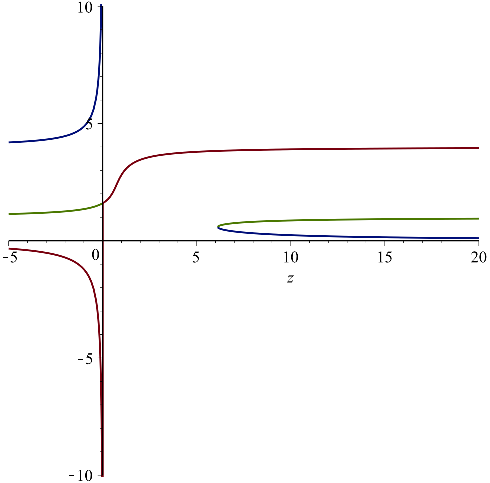
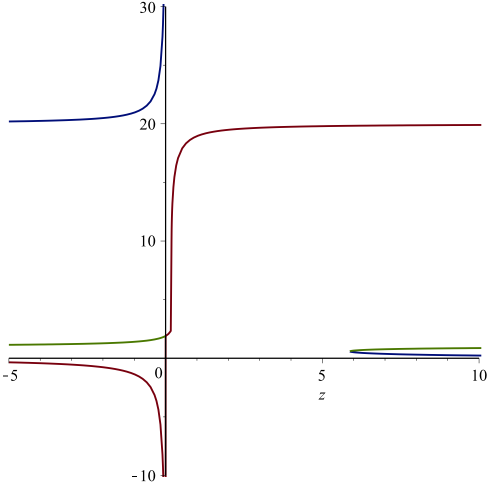
$0, b, c, d$ are the (real) roots of the discriminant of the cubic equation

$$\begin{aligned} & c_1^2 c_2^2 (c_1 - c_2)^2 z^4 - 6 c_1 c_2 (c_1 + c_2)(c_1 - c_2)^2 z^3 \\ + \; & (c_1^4 + 28 c_1^3 c_2 - 54 c_1^2 c_2^2 + 28 c_1 c_2^3 + c_2^4) z^2 - 4(c_1 + c_2)^3 z = 0. \end{aligned}$$



The functions ψ_0, ψ_1, ψ_2 for $c_1 = 1$, $c_2 = 4$ (interval is $[0, 6.12]$)



The functions ψ_0, ψ_1, ψ_2 for $c_1 = 1$, $c_2 = 20$ (intervals are $[0, 0.18] \cup [0.23, 5.87]$)

Multiple Laguerre II: asymptotics (two intervals)

Theorem (Lysov-Wielonsky)

Uniformly for z on compact subsets of $\mathbb{C} \setminus ([0,b] \cup [c,d])$

$$P_n(z) = -2^{\alpha+1/2} i \frac{\psi_0(z) - c_1)(\psi_0(z) - c_2)}{(z\psi_0(z))^\alpha \sqrt{D(\psi_0(z))}} e^{n(\lambda_0(z)-\ell_0)} \Big(1 + \mathcal{O}(1/n)\Big),$$

ℓ_0 *is a constant such that*

$$\lambda_0(z) = 2\log z + \ell_0 + \mathcal{O}(1/z),$$

and D is the polynomial

$$D(x) = 2(c_1+c_2)x^3 - (c_1^2 + 8c_1c_2 + c_2^2)x^2 + 4(c_1+c_2)c_1c_2x - 2c_1^2c_2^2.$$

Multiple Laguerre II: asymptotics

Theorem (Lysov-Wielonsky)

On the intervals $[\epsilon, b-\epsilon] \cup [c+\epsilon, d-\epsilon]$ *(with* $\epsilon > 0$*)*

$$P_n(x) = 2^{\alpha+1/2} x^{-\alpha} \Big(A(x) \sin[n \Im \lambda_0^+(x) + \varphi(x)] + \mathcal{O}(1/n) \Big) e^{n(\Re \lambda_0^+(x) - \ell_0)},$$

where

$$A(x) = 2 \left| \frac{(\psi_0^+(x) - c_1)(\psi_0^+(x) - c_2)}{(\psi_0^+)^\alpha(x) \sqrt{D(\psi_0^+(x))}} \right|,$$

and

$$\varphi(x) = \arg \frac{(\psi_0^+(x) - c_1)(\psi_0^+(x) - c_2)}{(\psi_0^+)^\alpha(x) \sqrt{D(\psi_0^+(x))}}.$$

Multiple Laguerre II: asymptotics

Theorem (Lysov-Wielonsky)

If z is in a neighborhood of d, then

$$P_n(z) = -2^{\alpha+1/2}\sqrt{\pi}\Big(n^{1/6}B_d(z)\mathrm{Ai}(n^{2/3}f_d(z))\big(1+\mathcal{O}(1/n)\big) + n^{-1/6}C_d(z)\mathrm{Ai}'(n^{2/3}f_d(z))\big(1+\mathcal{O}(1/n)\big)\Big)e^{n(\lambda_0(z)+\lambda_1(z)-2\ell_0)/2},$$

where Ai *is the Airy function, B_d and C_d are analytic functions in a neighborhood of d, and*

$$\frac{4}{3}[f_d(z)]^{3/2} = \lambda_1(z) - \lambda_0(z).$$

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Similar results for b and c.

Multiple Laguerre II: asymptotics

Theorem (Lysov-Wielonsky)

If z is in a neighborhood of 0, then

$$P_n(z) = -2^{\alpha+1/2} i\sqrt{\pi n} f_0(z)^{1/4} e^{-\alpha\pi/2}\Big(B_0(z)J_\alpha(2n(-f_0(z))^{1/2})(1+\mathcal{O}(1/n)) + C_0(z)J_\alpha'(2n(-f_0(z))^{1/2})(1+\mathcal{O}(1/n))\Big)e^{n(\lambda_0(z)+\lambda_2(z)-2\ell_0)/2},$$

where J_α is the Bessel function of order α, B_0 and C_0 are analytic functions in a neighborhood of 0, and

$$-4[f_0(z)]^{1/2} = \lambda_2(z) - \lambda_0(z) + 4\pi i.$$

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