Multiple Orthogonal Polynomials

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Plan of the course

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lecture 1: Definitions + basic properties

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Jacobi-Piñeiro + Jacobi-Angelesco

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lecture 4: Multiple Jacobi polynomials: Jacobi-Piñeiro + Jacobi-Angelesco

lecture 5: Riemann-Hilbert problem

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   This gives **Jacobi-Piñeiro polynomials**.

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2. Changing the parameter β to $\beta_1, \ldots, \beta_r$ (and $[-1,1]$ to $[0,1]$) Again Jacobi-Piñeiro polynomials, but $x \mapsto 1-x$.

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2. Changing the parameter β to $\beta_1, \ldots, \beta_r$ (and $[-1,1]$ to $[0,1]$) Again Jacobi-Piñeiro polynomials, but $x \mapsto 1-x$.
3. Use Jacobi weights on r disjoint intervals This gives **Jacobi-Angelesco polynomials**.

Jacobi-Piñeiro polynomials

Type II Jacobi-Piñeiro polynomials $P_{\vec{n}}^{\vec{\alpha},\beta}(x)$

$$\int_0^1 x^k P_{\vec{n}}^{\vec{\alpha},\beta}(x) x^{\alpha_j} (1-x)^\beta \, dx = 0, \qquad 0 \leq k \leq n_j - 1,$$

for $1 \leq j \leq r$.

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Rodrigues formula:

$$(-1)^{|\vec{n}|} \prod_{j=1}^r (|\vec{n}| + \alpha_j + \beta + 1)_{n_j} \; (1-x)^\beta P_{\vec{n}}^{\vec{\alpha},\beta}(x)$$
$$= \prod_{j=1}^r \left(x^{-\alpha_j} \frac{d^{n_j}}{dx^{n_j}} x^{n_j+\alpha_j} \right) (1-x)^{|\vec{n}|+\beta}.$$

Explicit formula:

$$P_{\vec{n}}^{\vec{\alpha},\beta}(x) = (-1)^{|\vec{n}|} \frac{n_1! \cdots n_r!}{\prod_{i=1}^{r}(|\vec{n}| + \alpha_j + \beta + 1)_{n_j}}$$

$$\sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} (-1)^{|\vec{k}|} \binom{|\vec{n}| + \beta}{|\vec{k}|} \prod_{j=1}^{r} \binom{n_j + \alpha_j + \sum_{i=1}^{j-1} k_i}{n_j - k_j} \binom{|\vec{k}|}{k_1, k_2, \ldots, k_r} x^{|\vec{k}|}(1-x)^{|\vec{n}|-|\vec{k}|}$$

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$$(-1)^{|\vec{n}|} (1-x)^{\beta} P_{\vec{n}}^{\vec{\alpha},\beta}(x) = \prod_{j=1}^r \frac{(\alpha_j + 1)_{n_j}}{(|\vec{n}| + \alpha_j + \beta + 1)_{n_j}}$$

$$ {}_{r+1}F_r \left(\begin{matrix} -|\vec{n}| - \beta, n_1 + \alpha_1 + 1, \ldots, n_r + \alpha_r + 1 \\ \alpha_1 + 1, \ldots, \alpha_r + 1 \end{matrix} \middle| \, x \right).$$

Jacobi-Piñeiro polynomials

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Limit relation:

$$\lim_{\beta \to \infty} \beta^{|\vec{n}|} P_{\vec{n}}^{\vec{\alpha},\beta}(x/\beta) = L_{\vec{n}}^{\vec{\alpha}}(x).$$

Recurrence relation:

$$xP_{\vec{n}}(x) = P_{\vec{n}+\vec{e}_k}(x) + b_{\vec{n},k}P_{\vec{n}}(x) + \sum_{j=1}^{r} a_{\vec{n},j}P_{\vec{n}-\vec{e}_j}(x)$$

$$a_{\vec{n},j} = \frac{n_j(n_j+\alpha_j)(|\vec{n}|+\beta)}{(|\vec{n}|+n_j+\alpha_j+\beta+1)(|\vec{n}|+n_j+\alpha_j+\beta)(|\vec{n}|+n_j+\alpha_j+\beta-1)} \times \prod_{i=1}^{r} \frac{|\vec{n}|+\alpha_i+\beta}{|\vec{n}|+n_i+\alpha_i+\beta} \prod_{i\neq j} \frac{n_j+\alpha_j-\alpha_i}{n_j-n_i+\alpha_j-\alpha_i}, \qquad 1 \leq j \leq r,$$

$$b_{\vec{n},k} = (|\vec{n}|+\beta+1)\frac{\prod_{j=1}^{r}(|\vec{n}|+\beta+\alpha_j+1)}{(|\vec{n}|+n_k+\alpha_k+\beta+2)\prod_{j\neq k}(|\vec{n}|+n_j+\alpha_j+\beta+1)} - (|\vec{n}|+\beta)\frac{\prod_{j=1}^{r}(|\vec{n}|+\beta+\alpha_j)}{\prod_{j=1}^{r}(|\vec{n}|+n_j+\beta+\alpha_j)}, \qquad 1 \leq k \leq r.$$

Jacobi-Piñeiro: differential properties

Raising operators:

$$\frac{d}{dx}\left(x^{\alpha_j+1}(1-x)^{\beta+1}P_{\vec{n}-\vec{e}_j}^{\vec{\alpha}+\vec{e}_j,\beta+1}(x)\right) = -(|\vec{n}|+\alpha_j+\beta+1)x^{\alpha_j}(1-x)^{\beta}P_{\vec{n}}^{\vec{\alpha},\beta}(x),$$

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Lowering operator:

$$\frac{d}{dx}P_{\vec{n}}^{\vec{\alpha},\beta}(x) = \sum_{j=1}^{r}\frac{\prod_{i=1}^{r}(n_i+\alpha_i-\alpha_j)}{\prod_{i=1,i\neq j}^{r}(\alpha_i-\alpha_j)}P_{\vec{n}-\vec{e}_j}^{\vec{\alpha}+\vec{e}_j,\beta+1}(x).$$

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Differential equation:

$$\left(\prod_{j=1}^{r}D_j\right)(1-x)^{\beta+1}DP_{\vec{n}}^{\vec{\alpha},\beta}(x) = -\sum_{j=1}^{r}\frac{\prod_{i=1}^{r}(n_i+\alpha_i-\alpha_j)}{\prod_{i=1,i\neq j}^{r}(\alpha_i-\alpha_j)}\left(\prod_{i\neq j}D_i\right)(1-x)^{\beta}P_{\vec{n}}^{\vec{\alpha},\beta}(x).$$

$$D = \frac{d}{dx}, \qquad D_j = x^{-\alpha_j}Dx^{\alpha_j+1}.$$

Integral representations:

$$(1-x)^{\beta} P_{\vec{n}}^{\vec{\alpha},\beta}(x) = \frac{\Gamma(n+\beta+1)}{\prod_{j=1}^{r}(|\vec{n}|+\alpha_j+\beta+1)_{n_j}} \frac{(-1)^{|\vec{n}|}}{2\pi i} \oint_{\Sigma} \frac{x^{-z-1}\Gamma(z+1)\prod_{j=1}^{r}(\alpha_j - z)_{n_j}}{\Gamma(z+n+\beta+2)}\, dz$$

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$$(-1)^{|\vec{n}|} Q_{\vec{n}}^{\vec{\alpha},\beta}(x) = \frac{\prod_{j=1}^r (n+\beta+\alpha_j)_{n_j}}{\Gamma(n+\beta)} (1-x)^\beta \frac{1}{2\pi i} \oint_\Gamma \frac{x^z \Gamma(z+n+\beta)}{\Gamma(z+1)\prod_{j=1}^r (\alpha_j - z)_{n_j}} dz$$

Γ is a contour enclosing $\alpha_j, \alpha_j+1, \ldots, \alpha_j+n_j-1$ for $1 \le j \le r$.

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$$(-1)^{|\vec{n}|}(1-x)^\beta \prod_{j=1}^r (|\vec{n}|+\alpha_j+\beta+1)_{n_j}\ P_{\vec{n}}^{\vec{\alpha},\beta}(x) = \frac{n_1!\cdots n_r!}{(2\pi i)^r}$$

$$\times \oint_\Sigma \cdots \oint_\Sigma \left(-\frac{x}{t_1\ldots t_r}\right)^{|\vec{n}|+\beta} \frac{t_1^{-\alpha_1-1}\cdots t_r^{-\alpha_r-1}}{(1-t_1)^{n_1+1}\cdots(1-t_r)^{n_r+1}}\, dt_1\cdots dt_r$$

Jacobi-Piñeiro: zero distribution

Theorem (E+J Coussement, VA)

Let $0 < x_{1,2n} < x_{2,2n} < \cdots < x_{2n,2n}$ *be the zeros of* $P_{n,n}^{(\alpha_1,\alpha_2),\beta}$. *Then*

$$\lim_{n\to\infty} \frac{1}{2n} \sum_{k=1}^{2n} f(x_{k,2n}) = \int_0^1 f(x) v_2(x)\, dx,$$

for every continuous function f *on* $[0,1]$, *where*

$$v_2(x) = \frac{\sqrt{3}}{4\pi} \frac{(1+\sqrt{1-x})^{1/3} + (1-\sqrt{1-x})^{1/3}}{x^{2/3}\sqrt{1-x}}.$$

Jacobi-Piñeiro: zero distribution

The proof uses the four-term recurrence relation

$$x p_n(x) = p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x) + d_n p_{n-2}(x),$$

$$p_{2n}(x) = P_{n,n}(x), \quad p_{2n+1}(x) = P_{n+1,n}(x)$$

where

$$\lim_{n\to\infty} b_n = 3, \quad \lim_{n\to\infty} c_n = 3, \quad \lim_{n\to\infty} d_n = 1.$$

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This gives ratio asymptotics

$$\lim_{n\to\infty} \frac{p_{n+1}(x)}{p_n(x)} = z(x), \qquad x \in \mathbb{C} \setminus [0,1],$$

where z is the solution of a **cubic equation**.

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$$\lim_{n\to\infty} \frac{1}{n} \frac{p_n'(x)}{p_n(x)} = \frac{z'(x)}{z(x)} = \int_0^1 \frac{v_2(t)}{x-t}\, dt, \qquad x \in \mathbb{C} \setminus [0,1].$$

Jacobi-Piñeiro: zero distribution

Theorem (Neuschel, VA)

Let $0 < x_{1,rn} < x_{2,rn} < \cdots < x_{rn,rn}$ *be the zeros of* $P_{n,n,\ldots,n}^{\vec{\alpha},\beta}$. *Then*

$$\lim_{n\to\infty} \frac{1}{rn} \sum_{k=1}^{rn} f(x_{k,rn}) = \int_0^1 f(x) v_r(x)\, dx,$$

where v_r *is given by*

$$\begin{aligned} v_r(x) &= \frac{r+1}{\pi} \frac{1}{|x'(\varphi)|} \\ &= \frac{r+1}{\pi x} \frac{\sin\varphi \sin r\varphi \sin(r+1)\varphi}{|(r+1)\sin r\varphi - r\sin(r+1)\varphi e^{i\varphi}|^2}, \end{aligned}$$

where

$$x = \frac{\big(\sin(r+1)\varphi\big)^{r+1}}{\sin\varphi(\sin r\varphi)^r}, \qquad 0 < \varphi < \frac{\pi}{r+1}.$$

Ratio asymptotics

Theorem (Van Assche)

Let $\vec{n} = (\lfloor q_1 n \rfloor, \ldots, \lfloor q_r n \rfloor)$, *where* $q_j > 0$ $(1 \leq j \leq r)$ *and* $q_1 + \cdots + q_r = 1$. *Suppose*

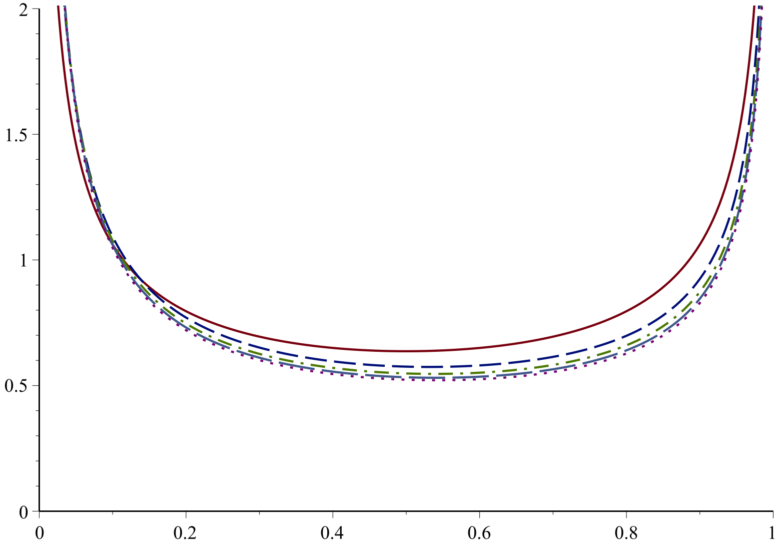
$$\lim_{n\to\infty} a_{\vec{n},j} = a_j, \quad \lim_{n\to\infty} b_{\vec{n},k} = b_j.$$

Then uniformly on compact sets of $\mathbb{C} \setminus \mathbb{R}$

$$\lim_{n\to\infty} \frac{P_{\vec{n}+\vec{e}_k}(x)}{P_{\vec{n}}(x)} = z - b_k,$$

where z *is the solution of the algebraic equation*

$$x - z = \sum_{j=1}^{r} \frac{a_j}{z - b_j}, \qquad \lim_{x\to\infty} z - x = 0.$$



$r = 1$ (solid), $r = 2$ (dash), $r = 3$ (dash dot), $r = 4$ (long dash), $r = 5$ (point)

Jacobi-Piñeiro polynomials: applications

If $\alpha_1 = \alpha_2 = \ldots = \alpha_r = \alpha$: limit case are still multiple orthogonal polynomials, but

$$\int_0^1 x^k P_{\vec{n}}^{\alpha,\beta}(x) x^\alpha (1-x)^\beta \, dx = 0, \qquad 0 \le k \le n_1 - 1,$$

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Normal indices when $n_1 \geq n_2 \geq n_3 \geq \ldots \geq n_r$.

Jacobi-Piñeiro polynomials: applications

For $\alpha = \beta = 0$ one has

$$(-1)^j \int_0^1 \frac{(\log x)^j}{1-x}\, dx = j!\boldsymbol{\zeta(j+1)}$$

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Hermite-Padé approximation to $(f_1, \ldots, f_r)$ with

$$f_j(z) = \frac{(-1)^{j-1}}{(j-1)!} \int_0^1 \frac{(\log x)^{j-1}}{z-x}\, dx$$

gives (for $z = 1$) rational approximants to $\zeta(1), \zeta(2), \ldots, \zeta(r)$.

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One problem arises: $\zeta(1) = \sum_{k=1}^{\infty} \frac{1}{k} = \infty$: additional interpolation conditions are needed, such as

$$A_{\vec{n},1}(1) = 0$$

for type I Hermite-Padé approximation.

Jacobi-Piñeiro polynomials: application

Theorem (Apéry, 1979)

$\zeta(3)$ *is irrational.*

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Infinitely many $\zeta(2n+1)$ are irrational.

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Theorem (Zudilin, 2002)

At least one of the numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational.

Jacobi-Angelesco polynomials

Introduced by V. Kalyagin in 1979 (for $r = 2$).

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Rodrigues formula:

$$(x-a)^\alpha x^\beta (1-x)^\gamma P_{n,n}^{\alpha,\beta,\gamma}(x) = \frac{(-1)^n}{(2n+\alpha+\beta+\gamma+1)_n} \frac{d^n}{dx^n} (x-a)^{n+\alpha} x^{n+\beta} (1-x)^{n+\gamma}.$$

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$$(x-a)^\alpha x^\beta (1-x)^\gamma P_{n,n+k}^{\alpha,\beta,\gamma}(x) = C_{n,k} \frac{d^n}{dx^n} \left((x-a)^{n+\alpha} x^{n+\beta} (1-x)^{n+\gamma} P_{0,k}^{\alpha+n,\beta+n,\gamma+n}(x) \right).$$

Jacobi-Angelesco polynomials: general r

Let $a_0 < a_1 < \cdots < a_r$ and $\beta > -1$ for $0 \leq j \leq r$.

$$w(x) = \prod_{j=0}^{r} |x - a_j|^{\beta_j}, \qquad x \in [a_0, a_r].$$

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$$\binom{|\vec{\beta}_j| + (r+1)n}{n} \prod_{j=0}^{r} (x - a_j)^{\beta_j} P_{n,n,\ldots,n}^{\vec{\beta}}(x) = \frac{1}{n!} \frac{d^n}{dx^n} \prod_{j=0}^{r} (x - a_j)^{n+\beta_j}.$$

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If $\vec{n} = (n, n, \ldots, n)$, then

$$\prod_{j=0}^{r} (x - a_j)^{\beta_j} P_{\vec{n}+\vec{m}}^{\vec{\beta}}(x) = C_{n,\vec{m}} \frac{d^n}{dx^n} \left(\prod_{j=0}^{r} (x - a_j)^{\beta_j + n} P_{\vec{m}}^{\vec{\beta}+\vec{n}}(x) \right).$$

Explicit expression: For $\vec{n} = (n, n, \ldots, n)$

$$\binom{|\vec{\beta}| + (r+1)n}{n} P_{\vec{n}}^{\vec{\beta}}(x) = \sum_{|k|=n} \binom{n+\beta_0}{k_0} \cdots \binom{n+\beta_r}{k_r} \prod_{j=0}^{r} (x-a_j)^{n-k_j}.$$

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Raising operator:

$$\frac{d}{dx} \left(\prod_{j=0}^{r} (x-a_j)^{\beta_j} P_{n,n,\ldots,n}^{\vec{\beta}}(x) \right) = (|\vec{\beta}|+rn) \prod_{j=0}^{r} (x-a_j)^{\beta_j - 1} P_{n+1,\ldots,n+1}^{\vec{\beta}-\vec{1}}(x).$$

Jacobi-Angelesco polynomials: zero distribution

For $P_{n,n}$ ($r = 2$) on the intervals $[a, 0] \cup [0, 1]$ (with $a < 0$): let

$$b = \frac{(a+1)^3}{9(a^2 - a + 1)}.$$

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The n zeros on the smallest interval push the n zeros on the larger interval away (pushing effect).

Jacobi-Angelesco polynomials: zero distribution

Equilibrium problem: let ν_1, ν_2 be probability measures with support on $[a, 0]$ and $[0, 1]$.

$$I(\nu_1) + I(\nu_2) + I(\nu_1, \nu_2) = \inf_{\mu_1,\mu_2} \Big(I(\mu_1) + I(\mu_2) + I(\mu_1, \mu_2) \Big)$$

where (mutual energy of μ_1 and μ_2)

$$I(\mu_1, \mu_2) = \int_0^1 \int_a^0 \log \frac{1}{|x-y|} \, d\mu_1(x) \, d\mu_2(y)$$

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If $x_{1,2n} < \ldots < x_{n,2n}$ are the zeros on $[a, 0]$ and $x_{n+1,2n} < \ldots < x_{2n,2n}$ are the zeros on $[0, 1]$ then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f(x_{j,2n}) = \int_a^0 f(x) \, d\nu_1(x), \qquad f \in C([a, 0]),$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=n+1}^{2n} g(x_{j,2n}) = \int_0^1 g(x) \, d\nu_2(x), \qquad g \in C([0, 1]).$$

Jacobi-Angelesco: asymptotics $[-1,0] \cup [0,1]$

Theorem (Kalyagin)

$$\binom{\alpha+\beta+\gamma+3n}{n} P_{n,n}(x) = \frac{1}{2\pi n} u_1^{-n} \Big(A(x) + o(1) \Big),$$

uniformly on compact sets of $\mathbb{C} \setminus [-1,1]$*, with* u_1 *a solution of*

$$u^3 + 3u^2 + \left(3 - \frac{27}{4}x^2\right) u + 1 = 0,$$

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$$u^3 + 3u^2 + \left(3 - \frac{27}{4}x^2\right) u + 1 = 0,$$

$$A(x) = \frac{i\sqrt{3}}{2} \frac{\left(\frac{3x}{2(1+u_1)} + 1\right)^{\alpha} \left(\frac{3x}{2(1+u_1)}\right)^{\beta} \left(\frac{3x}{2(1+u_1)} - 1\right)^{\gamma}}{(x+1)^{\alpha} x^{\beta} (x-1)^{\gamma}} \frac{(y^2-1)^{-1/4}(a-b)^{1/2}}{1+u_1}$$

with $y = 1/x$ and

$$a(y) = e^{4\pi i/3} \left(-y + \sqrt{y^2-1}\right)^{1/3}, \quad b(y) = e^{-4\pi i/3} \left(y - \sqrt{y^2-1}\right)^{1/3}.$$

Jacobi-Angelesco: asymptotics

Theorem (Kalyagin)

On the intervals $[-1+\epsilon, -\epsilon] \cup [\epsilon, 1-\epsilon]$ *(with* $\epsilon > 0$*)*

$$\binom{\alpha+\beta+\gamma+3n}{n} P_{n,n}(x) = \frac{1}{\sqrt{2\pi n}} |u_1|^{-n} |A(x)| \Big(\cos(n\theta - \varphi) + o(1) \Big),$$

where $\theta = \arg u_1(x)$ *and* $\varphi = \arg A(x)$.

Jacobi-Angelesco: asymptotics

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- If $a = -1$ then the asymptotic behavior near 0 is in terms of the **generalized Bessel function**

$$\lim_{n\to\infty} \frac{(-1)^n}{(\beta+1)_n} \binom{\alpha+\beta+\gamma+3n}{n} P_{n,n}(z/n^{3/2}) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(\beta+1)_{2k}\, k!}.$$

(Takata, 2005-2009).

Jacobi-Angelesco: application

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Type II Hermite-Padé approximation uses **Legendre-Angelesco polynomials** $P_{n,n} = P_{n,n}^{0,0,0}$

$$P_{n,n}(z)f_1(z) - Q_{n,n}(z) = \mathcal{O}\left(\frac{1}{z^{n+1}}\right),$$

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$$P_{n,n}(z)[2f_1(z) + f_2(z)] - [2Q_{n,n} + R_{n,n}(z)] = \mathcal{O}\left(\frac{1}{z^{n+1}}\right).$$

Evaluate at $z = i$ and estimate the error.

Jacobi-Angelesco: application

Definition

The measure of irrationality $\mu(x)$ of a real number x is

$$\mu(x) = \sup\{r > 0 : |x - \frac{a}{b}| < \frac{1}{b^r} \text{ has infinitely many solutions } (a,b) \in \mathbb{Z}^2\}.$$

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- If x is algebraic, then $\mu(x) = 2$ (Roth, 1955)
- There exist real numbers x for which $\mu(x) = +\infty$ (Liouville)

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$1, \log 2$ *and* π *are linearly independent over* $\mathbb{Q}$.

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Theorem (Salikhov, 2008)

$$\mu(\pi) \leq 7.6$$

References

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