Multiple Orthogonal Polynomials

Walter Van Assche

Summer school on OPSF, University of Kent

26–30 June, 2017

Plan of the course

Plan of the course

lecture 1: Definitions + basic properties

Plan of the course

lecture 1: Definitions + basic properties

lecture 2: Hermite-Padé, Multiple Hermite polynomials

Plan of the course

lecture 1: Definitions + basic properties

lecture 2: Hermite-Padé, Multiple Hermite polynomials

lecture 3: Multiple Laguerre polynomials (first and second kind)

Plan of the course

lecture 1: Definitions + basic properties

lecture 2: Hermite-Padé, Multiple Hermite polynomials

lecture 3: Multiple Laguerre polynomials (first and second kind)

lecture 4: Multiple Jacobi polynomials:
Jacobi-Angelesco + Jacobi-Piñeiro

Plan of the course

lecture 1: Definitions + basic properties

lecture 2: Hermite-Padé, Multiple Hermite polynomials

lecture 3: Multiple Laguerre polynomials (first and second kind)

lecture 4: Multiple Jacobi polynomials:
Jacobi-Angelesco + Jacobi-Piñeiro

lecture 5: Riemann-Hilbert problem

Riemann-Hilbert problem for MOPS

Fokas, Its and Kitaev[1] formulated a Riemann-Hilbert problem (for 2×2 matrices) that characterizes orthogonal polynomials.

[1] A.S. Fokas, A. Its, A.V. Kitaev, *The isomonodromy approach to matrix models in 2D quantum gravity*, Comm. Math. Phys. **147** (1992), no. 2, 395–430

Riemann-Hilbert problem for MOPS

Fokas, Its and Kitaev[1] formulated a Riemann-Hilbert problem (for 2×2 matrices) that characterizes orthogonal polynomials.

There is a Riemann-Hilbert problem (for $(r+1) \times (r+1)$ matrices) that characterizes multiple orthogonal polynomials (W. Van Assche, J.S. Geronimo, A.B.J. Kuijlaars, 2001). Suppose that $d\mu_j(x) = w_j(x)\,dx$ $(1 \leq j \leq r)$

[1] A.S. Fokas, A. Its, A.V. Kitaev, *The isomonodromy approach to matrix models in 2D quantum gravity*, Comm. Math. Phys. **147** (1992), no. 2, 395–430

Riemann-Hilbert problem for MOPS

Fokas, Its and Kitaev[1] formulated a Riemann-Hilbert problem (for 2×2 matrices) that characterizes orthogonal polynomials.

There is a Riemann-Hilbert problem (for $(r+1) \times (r+1)$ matrices) that characterizes multiple orthogonal polynomials (W. Van Assche, J.S. Geronimo, A.B.J. Kuijlaars, 2001). Suppose that $d\mu_j(x) = w_j(x)\,dx$ $(1 \leq j \leq r)$

Find $Y = \mathbb{C} \to \mathbb{C}^{(r+1)\times(r+1)}$ for which

1. Y is analytic on $\mathbb{C} \setminus \mathbb{R}$.

[1] A.S. Fokas, A. Its, A.V. Kitaev, *The isomonodromy approach to matrix models in 2D quantum gravity*, Comm. Math. Phys. **147** (1992), no. 2, 395–430

Riemann-Hilbert problem for MOPS

2. The boundary values $Y_\pm(x) = \lim_{\epsilon\to 0+} Y(x \pm i\epsilon)$ exist for $x \in \mathbb{R}$ and satisfy

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w_1(x) & w_2(x) & \cdots & w_r(x) \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \qquad x \in \mathbb{R}.$$

Riemann-Hilbert problem for MOPS

3. Y behaves near infinity as

$$Y(z) = \Big(I + \mathcal{O}(1/z)\Big) \begin{pmatrix} z^{|\vec{n}|} & & & & 0 \\ & z^{-n_1} & & & \\ & & z^{-n_2} & & \\ & & & \ddots & \\ 0 & & & & z^{-n_r} \end{pmatrix}, \qquad z \to \infty.$$

Riemann-Hilbert problem for MOPS

3. Y behaves near infinity as

$$Y(z) = \Big(I + \mathcal{O}(1/z)\Big) \begin{pmatrix} z^{|\vec{n}|} & & & & 0 \\ & z^{-n_1} & & & \\ & & z^{-n_2} & & \\ & & & \ddots & \\ 0 & & & & z^{-n_r} \end{pmatrix}, \qquad z \to \infty.$$

4. [some condition near the endpoints of the supports of $\mu_1, \ldots, \mu_r$]

Riemann-Hilbert problem for MOPS

Theorem (VA, Geronimo, Kuijlaars)

If $\vec{n}$ and $\vec{n} - \vec{e}_j$ $(1 \leq j \leq r)$ are normal indices, then

$$Y(z) = \begin{pmatrix} P_{\vec{n}}(z) & \frac{1}{2\pi i} \int \frac{P_{\vec{n}}(x) w_1(x)}{x-z}\, dx & \cdots & \frac{1}{2\pi i} \int \frac{P_{\vec{n}}(x) w_r(x)}{x-z}\, dx \\ -2\pi i \gamma_1 P_{\vec{n}-\vec{e}_1}(z) & -\gamma_1 \int \frac{P_{\vec{n}-\vec{e}_1}(x) w_1(x)}{x-z}\, dx & \cdots & -\gamma_1 \int \frac{P_{\vec{n}-\vec{e}_1}(x) w_r(x)}{x-z}\, dx \\ \vdots & \vdots & \cdots & \vdots \\ -2\pi i \gamma_r P_{\vec{n}-\vec{e}_r}(z) & -\gamma_r \int \frac{P_{\vec{n}-\vec{e}_r}(x) w_1(x)}{x-z}\, dx & \cdots & -\gamma_r \int \frac{P_{\vec{n}-\vec{e}_r}(x) w_r(x)}{x-z}\, dx \end{pmatrix}$$

where

$$\frac{1}{\gamma_j} = \frac{1}{\gamma_j(\vec{n})} = \int x^{n_j - 1} P_{\vec{n}-\vec{e}_j} w_j(x)\, dx, \qquad 1 \leq j \leq r.$$

Riemann-Hilbert problem for MOPS

Theorem (VA, Geronimo, Kuijlaars)

If $\vec{n}$ and $\vec{n} - \vec{e}_j$ $(1 \leq j \leq r)$ are normal indices, then

$$Y^{-T}(z) = \begin{pmatrix} \int \frac{Q_{\vec{n}}(x)}{z-x}\,dx & 2\pi i A_{\vec{n},1}(z) & \cdots & 2\pi i A_{\vec{n},r}(z) \\ \frac{c_1}{2\pi i} \int \frac{Q_{\vec{n}+\vec{e}_1}(x)}{z-x}\,dx & c_1 A_{\vec{n}+\vec{e}_1,1}(z) & \cdots & c_1 A_{\vec{n}+\vec{e}_1,r}(z) \\ \vdots & \vdots & \cdots & \vdots \\ \frac{c_r}{2\pi i} \int \frac{Q_{\vec{n}+\vec{e}_r}(x)}{z-x}\,dx & c_r A_{\vec{n}+\vec{e}_r,1}(z) & \cdots & c_r A_{\vec{n}+\vec{e}_r,r}(z) \end{pmatrix}$$

where

$$A_{\vec{n}+\vec{e}_j}(z) = \frac{z^{n_j}}{c_j(\vec{n})} + \text{ lower order terms.}$$

Nikishin system

Introduced by E.M. Nikishin in 1980.

Nikishin system

Introduced by E.M. Nikishin in 1980.

- μ_2 is absolutely continuous with respect to μ_1

Nikishin system

Introduced by E.M. Nikishin in 1980.

- μ_2 is absolutely continuous with respect to μ_1
- The Radon-Nikodym derivative is

$$\frac{d\mu_2}{d\mu_1} = w(x) = \int_c^d \frac{d\sigma(t)}{x-t},$$

Nikishin system

Introduced by E.M. Nikishin in 1980.

- μ_2 is absolutely continuous with respect to μ_1
- The Radon-Nikodym derivative is

$$\frac{d\mu_2}{d\mu_1} = w(x) = \int_c^d \frac{d\sigma(t)}{x-t},$$

- $[c,d]$ is disjoint from $[a,b]$.

Nikishin system

Introduced by E.M. Nikishin in 1980.

- μ_2 is absolutely continuous with respect to μ_1
- The Radon-Nikodym derivative is

$$\frac{d\mu_2}{d\mu_1} = w(x) = \int_c^d \frac{d\sigma(t)}{x-t},$$

- $[c,d]$ is disjoint from $[a,b]$.

We will take $c < d < a < b$.

Multiple orthogonal polynomials for a Nikishin system

We consider the following case:

$$d\mu_1(x) = (x-a)^\alpha (b-x)^\beta h_1(x)\, dx = w_1(x)\, dx, \qquad x \in [a,b],$$

$$d\sigma(t) = (t-c)^\gamma (d-t)^\delta h_2(t)\, dt = w_2(t)\, dt \qquad t \in [c,d],$$

where h_1 is analytic in a neighborhood of $[a,b]$ and h_2 is analytic in a neighborhood of $[c,d]$.

Multiple orthogonal polynomials for a Nikishin system

We consider the following case:

$$d\mu_1(x) = (x-a)^\alpha (b-x)^\beta h_1(x)\, dx = w_1(x)\, dx, \qquad x \in [a,b],$$

$$d\sigma(t) = (t-c)^\gamma (d-t)^\delta h_2(t)\, dt = w_2(t)\, dt \qquad t \in [c,d],$$

where h_1 is analytic in a neighborhood of $[a,b]$ and h_2 is analytic in a neighborhood of $[c,d]$.

$$\int_a^b x^k \Big(A_{n,m}(x) + w(x) B_{n,m}(x) \Big) w_1(x)\, dx = 0, \qquad 0 \le k \le n+m-2,$$

$$w(x) = \int_c^d \frac{w_2(t)}{x-t}\, dt.$$

Riemann-Hilbert problem

We use the Riemann-Hilbert problem for $X = Y^{-T}$

Riemann-Hilbert problem

We use the Riemann-Hilbert problem for $X = Y^{-T}$

Find $X : \mathbb{C} \to \mathbb{C}^{3\times 3}$ such that

Riemann-Hilbert problem

We use the Riemann-Hilbert problem for $X = Y^{-T}$

Find $X : \mathbb{C} \to \mathbb{C}^{3\times 3}$ such that

1. X is analytic in $\mathbb{C} \setminus [a, b]$.

Riemann-Hilbert problem

We use the Riemann-Hilbert problem for $X = Y^{-T}$

Find $X : \mathbb{C} \to \mathbb{C}^{3\times 3}$ such that

1. X is analytic in $\mathbb{C} \setminus [a,b]$.
2. $\lim_{\epsilon\to 0+} X(x \pm i\epsilon) = X_\pm(x)$ exists for $x \in (a,b)$ and

$$X_+(x) = X_-(x) \begin{pmatrix} 1 & 0 & 0 \\ -2\pi i w_1(x) & 1 & 0 \\ -2\pi i w(x) w_1(x) & 0 & 1 \end{pmatrix}, \qquad x \in (a,b).$$

Riemann-Hilbert problem

3. Near infinity one has

$$X(z) = \left(\mathbb{I} + \mathcal{O}(1/z)\right) \begin{pmatrix} z^{-n-m} & 0 & 0 \\ 0 & z^n & 0 \\ 0 & 0 & z^m \end{pmatrix}, \qquad z \to \infty.$$

Riemann-Hilbert problem

3. Near infinity one has

$$X(z) = \left(\mathbb{I} + \mathcal{O}(1/z)\right) \begin{pmatrix} z^{-n-m} & 0 & 0 \\ 0 & z^n & 0 \\ 0 & 0 & z^m \end{pmatrix}, \qquad z \to \infty.$$

4. Near a the behavior is

$$X(z) = \begin{pmatrix} \mathcal{O}(r_a(z)) & \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(r_a(z)) & \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(r_a(z)) & \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix}, \qquad z \to a,$$

where

$$r_a(z) = \begin{cases} |z-a|^\alpha, & -1 < \alpha < 0, \\ \log|z-a|, & \alpha = 0, \\ 1, & \alpha > 0. \end{cases}$$

Riemann-Hilbert problem

3. Near infinity one has

$$X(z) = \left(\mathbb{I} + \mathcal{O}(1/z)\right) \begin{pmatrix} z^{-n-m} & 0 & 0 \\ 0 & z^n & 0 \\ 0 & 0 & z^m \end{pmatrix}, \qquad z \to \infty.$$

4. Near b the behavior is

$$X(z) = \begin{pmatrix} \mathcal{O}(r_b(z)) & \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(r_b(z)) & \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(r_b(z)) & \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix}, \qquad z \to b,$$

where

$$r_b(z) = \begin{cases} |z-b|^\beta, & -1 < \beta < 0, \\ \log|z-b|, & \beta = 0, \\ 1, & \beta > 0. \end{cases}$$

Riemann-Hilbert problem

The solution is

$$X(z) = \begin{pmatrix} \displaystyle\int_a^b \frac{A_{n,m}(x) + w(x)B_{n,m}(x)}{z-x} w_1(x)\,dx & A_{n,m}(z) & B_{n,m}(z) \\ \displaystyle c_1 \int_a^b \frac{A_{n+1,m}(x) + w(x)B_{n+1,m}(x)}{z-x} w_1(x)\,dx & c_1 A_{n+1,m}(z) & c_1 B_{n+1,m}(z) \\ \displaystyle c_2 \int_a^b \frac{A_{n,m+1}(x) + w(x)B_{n,m+1}(x)}{z-x} w_1(x)\,dx & c_2 A_{n,m+1}(z) & c_2 B_{n,m+1}(z) \end{pmatrix}$$

with $c_1 = c_1(n,m)$ and $c_2 = c_2(n,m)$ such that

$$c_1 A_{n+1,m}(z) = z^n + \text{ lower order terms,}$$
$$c_2 B_{n,m+1}(z) = z^m + \text{ lower order terms.}$$

Riemann-Hilbert analysis

The idea is to transform this Riemann-Hilbert problem for X to a Riemann-Hilbert problem for R that behaves nicely, uniformly in $\mathbb{C}$,

$$\lim_{n,m\to\infty} R(z) = \mathbb{I}.$$

Then undo the transformations for the asymptotic behavior of X.

The idea is to transform this Riemann-Hilbert problem for X to a Riemann-Hilbert problem for R that behaves nicely, uniformly in $\mathbb{C}$,

$$\lim_{n,m\to\infty} R(z) = \mathbb{I}.$$

Then undo the transformations for the asymptotic behavior of X.

First transformation:

$$U(z) = X(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \displaystyle\int_c^d \frac{w_2(t)}{z-t}\,dt & 1 \end{pmatrix}$$

Riemann-Hilbert analysis

The idea is to transform this Riemann-Hilbert problem for X to a Riemann-Hilbert problem for R that behaves nicely, uniformly in $\mathbb{C}$,

$$\lim_{n,m\to\infty} R(z) = \mathbb{I}.$$

Then undo the transformations for the asymptotic behavior of X.

First transformation:

$$U(z) = X(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \displaystyle\int_c^d \frac{w_2(t)}{z-t}\, dt & 1 \end{pmatrix}$$

This brings in the measure $d\sigma(t) = w_2(t)\, dt$ on the interval $[c, d]$

Riemann-Hilbert problem for U

1. U is analytic in $\mathbb{C} \setminus ([a, b] \cup [c, d])$

Riemann-Hilbert problem for U

1. U is analytic in $\mathbb{C} \setminus ([a,b] \cup [c,d])$
2. Jumps

$$U_+(x) = U_-(x) \begin{pmatrix} 1 & 0 & 0 \\ -2\pi i w_1(x) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad x \in (a,b),$$

$$U_+(x) = U_-(x) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2\pi i w_2(x) & 1 \end{pmatrix}, \qquad x \in (c,d).$$

Riemann-Hilbert problem for U

3. Asymptotic behavior (here one needs $m \leq n$)

$$U(z) = \left(\mathbb{I} + \mathcal{O}(1/z)\right) \begin{pmatrix} z^{-n-m} & 0 & 0 \\ 0 & z^n & 0 \\ 0 & 0 & z^m \end{pmatrix}, \qquad z \to \infty.$$

Riemann-Hilbert problem for U

3. Asymptotic behavior (here one needs $m \leq n$)

$$U(z) = \left(\mathbb{I} + \mathcal{O}(1/z)\right) \begin{pmatrix} z^{-n-m} & 0 & 0 \\ 0 & z^n & 0 \\ 0 & 0 & z^m \end{pmatrix}, \qquad z \to \infty.$$

4. Behavior near a

$$U(z) = \begin{pmatrix} \mathcal{O}(r_a(z)) & \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(r_a(z)) & \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(r_a(z)) & \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix}, \qquad z \to a,$$

Riemann-Hilbert problem for U

3. Asymptotic behavior (here one needs $m \leq n$)

$$U(z) = \left(\mathbb{I} + \mathcal{O}(1/z)\right) \begin{pmatrix} z^{-n-m} & 0 & 0 \\ 0 & z^n & 0 \\ 0 & 0 & z^m \end{pmatrix}, \qquad z \to \infty.$$

4. Behavior near b

$$U(z) = \begin{pmatrix} \mathcal{O}(r_b(z)) & \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(r_b(z)) & \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(r_b(z)) & \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix}, \qquad z \to b,$$

Riemann-Hilbert problem for U

3. Asymptotic behavior (here one needs $m \leq n$)

$$U(z) = \left(\mathbb{I} + \mathcal{O}(1/z)\right) \begin{pmatrix} z^{-n-m} & 0 & 0 \\ 0 & z^n & 0 \\ 0 & 0 & z^m \end{pmatrix}, \qquad z \to \infty.$$

4. Behavior near c

$$U(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(r_c(z)) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(r_c(z)) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(r_c(z)) & \mathcal{O}(1) \end{pmatrix}, \qquad z \to c,$$

Riemann-Hilbert problem for U

3. Asymptotic behavior (here one needs $m \leq n$)

$$U(z) = \left(\mathbb{I} + \mathcal{O}(1/z)\right) \begin{pmatrix} z^{-n-m} & 0 & 0 \\ 0 & z^n & 0 \\ 0 & 0 & z^m \end{pmatrix}, \qquad z \to \infty.$$

4. Behavior near d

$$U(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(r_d(z)) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(r_d(z)) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(r_d(z)) & \mathcal{O}(1) \end{pmatrix}, \qquad z \to d,$$

Equilibrium problem:

$$2I(\nu_1)+2q_1^2 I(\nu_2)-2q_1 I(\nu_1,\nu_2) = \inf_{\mu_1,\mu_2}\Big(2I(\mu_1)+2q_1^2 I(\mu_2)-2q_1 I(\mu_1,\mu_2)\Big),$$

μ_1, μ_2 are probability measures, $\mathrm{supp}(\mu_1) \subset [a,b]$, $\mathrm{supp}(\mu_2) \subset [c,d]$ and $q_1 = \frac{m}{n+m}$

$$I(\mu_1,\mu_2) = \int_c^d \int_a^b \log\frac{1}{|x-y|}\, d\mu_1(x)\, d\mu_2(x).$$

Vector equilibrium problem

Equilibrium problem:

$$2I(\nu_1)+2q_1^2I(\nu_2)-2q_1I(\nu_1,\nu_2) = \inf_{\mu_1,\mu_2}\Big(2I(\mu_1)+2q_1^2I(\mu_2)-2q_1I(\mu_1,\mu_2)\Big),$$

μ_1, μ_2 are probability measures, $\text{supp}(\mu_1) \subset [a, b]$, $\text{supp}(\mu_2) \subset [c, d]$ and $q_1 = \frac{m}{n+m}$

$$I(\mu_1, \mu_2) = \int_c^d \int_a^b \log \frac{1}{|x-y|}\, d\mu_1(x)\, d\mu_2(x).$$

- ν_1 gives the asymptotic distribution of the zeros of $A_{n,m}(x) + w(x)B_{n,m}(x)$ on $[a, b]$

Equilibrium problem:

$$2I(\nu_1)+2q_1^2I(\nu_2)-2q_1I(\nu_1,\nu_2) = \inf_{\mu_1,\mu_2}\Big(2I(\mu_1)+2q_1^2I(\mu_2)-2q_1I(\mu_1,\mu_2)\Big),$$

μ_1, μ_2 are probability measures, $\text{supp}(\mu_1) \subset [a,b]$, $\text{supp}(\mu_2) \subset [c,d]$ and $q_1 = \frac{m}{n+m}$

$$I(\mu_1,\mu_2) = \int_c^d \int_a^b \log \frac{1}{|x-y|}\, d\mu_1(x)\, d\mu_2(x).$$

- ν_1 gives the asymptotic distribution of the zeros of $A_{n,m}(x) + w(x)B_{n,m}(x)$ on $[a,b]$
- ν_2 gives the asymptotic distribution of the zeros of $B_{n,m}$ on $[c,d]$.

Variational conditions:

$$\begin{aligned} 2U(x;\nu_1) - q_1 U(x;\nu_2) &= \ell_1, \qquad & x \in [a,b], \\ 2q_1 U(x;\nu_2) - U(x;\nu_1) &= \ell_2, \qquad & x \in [c,d], \end{aligned}$$

where

$$U(x;\mu) = \int \log \frac{1}{|x-y|}\, d\mu(y).$$

Riemann-Hilbert problem: g-functions

Introduce

$$g_1(z) = \int_a^b \log(z-x)\, d\nu_1(x), \quad g_2(z) = \int_c^d \log(z-t)\, d\nu_2(t).$$

Riemann-Hilbert problem: g-functions

Introduce

$$g_1(z) = \int_a^b \log(z-x)\,d\nu_1(x), \quad g_2(z) = \int_c^d \log(z-t)\,d\nu_2(t).$$

$$g_1^{\pm}(x) = \begin{cases} -U(x;\nu_1), & x > b, \\ -U(x,\nu_1) \pm i\pi, & x < a, \\ -U(x;\nu_1) \pm i\pi\varphi_1(x), & a < x < b, \end{cases} \qquad \varphi_1(x) = \int_x^b d\nu_1(t),$$

$$g_2^{\pm}(x) = \begin{cases} -U(x;\nu_2), & x > d, \\ -U(x,\nu_2) \pm i\pi, & x < c, \\ -U(x;\nu_2) \pm i\pi\varphi_2(x), & c < x < d, \end{cases} \qquad \varphi_2(x) = \int_x^d d\nu_2(t),$$

Riemann-Hilbert problem: second transformation

Normalizing the Riemann-Hilbert problem:

$$V(z) = LU(z) \begin{pmatrix} e^{(n+m)g_1(z)} & 0 & 0 \\ 0 & e^{-(n+m)g_1(z)+mg_2(z)} & 0 \\ 0 & 0 & e^{-mg_2(z)} \end{pmatrix} L^{-1},$$

Riemann-Hilbert problem: second transformation

Normalizing the Riemann-Hilbert problem:

$$V(z) = LU(z) \begin{pmatrix} e^{(n+m)g_1(z)} & 0 & 0 \\ 0 & e^{-(n+m)g_1(z)+mg_2(z)} & 0 \\ 0 & 0 & e^{-mg_2(z)} \end{pmatrix} L^{-1},$$

where

$$L = L(n,m) = \begin{pmatrix} e^{-\frac{n+m}{3}(2\ell_1+\ell_2)} & 0 & 0 \\ 0 & e^{\frac{n+m}{3}(\ell_1-\ell_2)} & 0 \\ 0 & 0 & e^{\frac{n+m}{3}(\ell_1+2\ell_2)} \end{pmatrix}.$$

Riemann-Hilbert problem for V

1. V is analytic on $\mathbb{C} \setminus ([a, b] \cup [c, d])$.

Riemann-Hilbert problem for V

1. V is analytic on $\mathbb{C} \setminus ([a,b] \cup [c,d])$.
2. Asymptotic behavior in normalized

$$V(z) = \mathbb{I} + \mathcal{O}(1/z), \qquad z \to \infty.$$

Riemann-Hilbert problem for V

1. V is analytic on $\mathbb{C} \setminus ([a,b] \cup [c,d])$.
2. Asymptotic behavior in normalized

$$V(z) = \mathbb{I} + \mathcal{O}(1/z), \qquad z \to \infty.$$

3. Oscillatory jumps on (a,b) and (c,d)

$$V_+(x) = V_-(x) \begin{pmatrix} e^{2\pi i(n+m)\varphi_1(x)} & 0 & 0 \\ -2\pi i w_1(x) & e^{-2\pi i(n+m)\varphi_1(x)} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad x \in (a,b),$$

$$V_+(x) = V_-(x) \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i m\varphi_2(x)} & 0 \\ 0 & -2\pi i w_2(x) & e^{-2\pi i m\varphi_2(x)} \end{pmatrix}, \qquad x \in (c,d).$$

Riemann-Hilbert problem for V

1. V is analytic on $\mathbb{C} \setminus ([a,b] \cup [c,d])$.
2. Asymptotic behavior in normalized

$$V(z) = \mathbb{I} + \mathcal{O}(1/z), \qquad z \to \infty.$$

3. Oscillatory jumps on (a,b) and (c,d)

$$V_+(x) = V_-(x) \begin{pmatrix} e^{2\pi i(n+m)\varphi_1(x)} & 0 & 0 \\ -2\pi i w_1(x) & e^{-2\pi i(n+m)\varphi_1(x)} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad x \in (a,b),$$

$$V_+(x) = V_-(x) \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i m\varphi_2(x)} & 0 \\ 0 & -2\pi i w_2(x) & e^{-2\pi i m\varphi_2(x)} \end{pmatrix}, \qquad x \in (c,d).$$

4. Behavior near a, b, c, d under control.

Steepest descent for oscillatory Riemann-Hilbert problems

Deift and Zhou (1993) developed a method for obtaining the asymptotic behavior (in our case for $n, m \to \infty$) for oscillatory Riemann-Hilbert problems. It starts with factoring the oscillatory jump.

Steepest descent for oscillatory Riemann-Hilbert problems

$$\begin{pmatrix} e^{2\pi i(n+m)\varphi_1(x)} & 0 & 0 \\ -2\pi i w_1(x) & e^{-2\pi i(n+m)\varphi_1(x)} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \Phi_1^{n+m} & 0 & 0 \\ -v_1 & \bar{\Phi}_1^{n+m} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -\Phi_1^{n+m}/v_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1/v_1 & 0 \\ -v_1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\bar{\Phi}_1^{n+m}/v_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Steepest descent for oscillatory Riemann-Hilbert problems

$$\begin{pmatrix} e^{2\pi i(n+m)\varphi_1(x)} & 0 & 0 \\ -2\pi i w_1(x) & e^{-2\pi i(n+m)\varphi_1(x)} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \Phi_1^{n+m} & 0 & 0 \\ -v_1 & \bar{\Phi}_1^{n+m} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

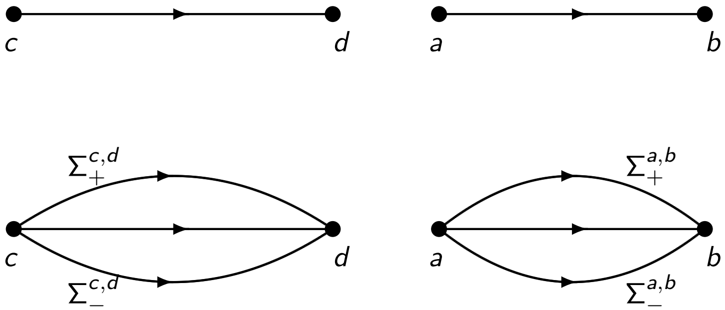
$$= \begin{pmatrix} 1 & -\Phi_1^{n+m}/v_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1/v_1 & 0 \\ -v_1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\bar{\Phi}_1^{n+m}/v_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i m\varphi_2(x)} & 0 \\ 0 & -2\pi i w_2(x) & e^{-2\pi i m\varphi_2(x)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Phi_2^m & 0 \\ 0 & -v_2 & \bar{\Phi}_2^m \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\Phi_2^m/v_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1/v_2 \\ 0 & -v_2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\bar{\Phi}^m/v_2 \\ 0 & 0 & 1 \end{pmatrix}$$

Opening the lenses

Opening the lenses



Riemann-Hilbert problem: third transformation

$$S(z) = \begin{cases} V(z), & \text{outside the lenses} \\ V(z) \begin{pmatrix} 1 & \Phi_1^{-(n+m)}/v_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \text{inside the } [a,b]\text{-lens, upper part} \\ V(z) \begin{pmatrix} 1 & -\Phi_1^{-(n+m)}/v_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \text{inside the } [a,b]\text{-lens, lower part} \\ V(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \Phi_2^{-m}/v_2 \\ 0 & 0 & 1 \end{pmatrix}, & \text{inside the } [c,d]\text{-lens, upper part} \\ V(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\Phi_2^{-m}/v_2 \\ 0 & 0 & 1 \end{pmatrix}, & \text{inside the } [c,d]\text{-lens, lower part} \end{cases}$$

Riemann-Hilbert problem for S

1. S is analytic in $\mathbb{C} \setminus ([a,b] \cup [c,d] \cup \Sigma^{a,b} \cup \Sigma^{c,d})$.

Riemann-Hilbert problem for S

1. S is analytic in $\mathbb{C} \setminus ([a,b] \cup [c,d] \cup \Sigma^{a,b} \cup \Sigma^{c,d})$.
2. S is normalized near infinity: $S(z) = \mathbb{I} + \mathcal{O}(1/z)$.

Riemann-Hilbert problem for S

1. S is analytic in $\mathbb{C} \setminus ([a,b] \cup [c,d] \cup \Sigma^{a,b} \cup \Sigma^{c,d})$.
2. S is normalized near infinity: $S(z) = \mathbb{I} + \mathcal{O}(1/z)$.
3. Jumps

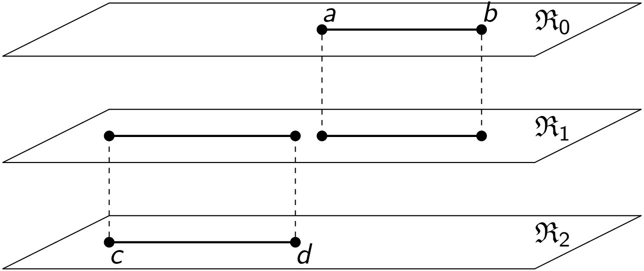
$$S_+(z) = \begin{cases} S_-(z) \begin{pmatrix} 1 & -\Phi_1^{-(n+m)}/v_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \Sigma^{a,b}, \\ S_-(z) \begin{pmatrix} 0 & 1/v_1 & 0 \\ -v_1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in (a,b), \\ S_-(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\Phi_2^{-m}/v_2 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \Sigma^{c,d}, \\ S_-(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1/v_2 \\ 0 & -v_2 & 0 \end{pmatrix}, & z \in (c,d). \end{cases}$$

Riemann surface $\mathfrak{R}$

We need to extend Φ_1 (defined originally on $[a, b]$) and Φ_2 (defined originally on $[c, d]$) to the complex plane.

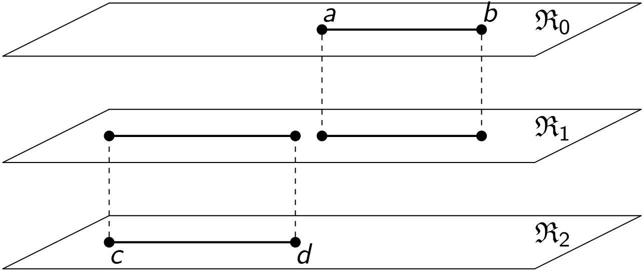
Riemann surface $\mathfrak{R}$

We need to extend Φ_1 (defined originally on $[a, b]$) and Φ_2 (defined originally on $[c, d]$) to the complex plane.
These functions live more naturally on a Riemann surface with three sheets.



Riemann surface $\mathfrak{R}$

We need to extend Φ_1 (defined originally on $[a, b]$) and Φ_2 (defined originally on $[c, d]$) to the complex plane.
These functions live more naturally on a Riemann surface with three sheets.



$$\Phi_1^{\pm}(x) = e^{\pm 2\pi i \varphi_1(x)}, \qquad \Phi_2^{\pm}(x) = e^{\pm 2\pi i \varphi_2(x)}$$

Oscillatory jumps $\to$ exponentially small jumps

Claim:

$$\begin{cases} |\Phi_1(z)| > 1, & z \in \Sigma^{a,b}, \\ |\Phi_2(z)| > 1, & z \in \Sigma^{c,d}. \end{cases}$$

Oscillatory jumps $\to$ exponentially small jumps

Claim:

$$\begin{cases} |\Phi_1(z)| > 1, & z \in \Sigma^{a,b}, \\ |\Phi_2(z)| > 1, & z \in \Sigma^{c,d}. \end{cases}$$

As $n, m \to \infty$ the jumps on $\Sigma^{a,b}$ and $\Sigma^{c,d}$ tend to $\mathbb{I}$ exponentially fast.

Oscillatory jumps $\to$ exponentially small jumps

Claim:

$$\begin{cases} |\Phi_1(z)| > 1, & z \in \Sigma^{a,b}, \\ |\Phi_2(z)| > 1, & z \in \Sigma^{c,d}. \end{cases}$$

As $n, m \to \infty$ the jumps on $\Sigma^{a,b}$ and $\Sigma^{c,d}$ tend to $\mathbb{I}$ exponentially fast.

The main contribution to the Riemann-Hilbert matrix S will be the jumps on $[a, b]$ and $[c, d]$.

Global parametrix

We ignore the jumps that tend to $\mathbb{I}$.

Global parametrix

We ignore the jumps that tend to $\mathbb{I}$.

The main contribution to S is the matrix N with satisfies the Riemann-Hilbert problem

1. N is analytic in $\mathbb{C} \setminus ([a,b] \cup [c,d])$.

Global parametrix

We ignore the jumps that tend to $\mathbb{I}$.

The main contribution to S is the matrix N with satisfies the Riemann-Hilbert problem

1. N is analytic in $\mathbb{C} \setminus ([a,b] \cup [c,d])$.
2. N is normalized near infinity: $N(z) = \mathbb{I} + \mathcal{O}(1/z)$.

Global parametrix

We ignore the jumps that tend to $\mathbb{I}$.

The main contribution to S is the matrix N with satisfies the Riemann-Hilbert problem

1. N is analytic in $\mathbb{C} \setminus ([a, b] \cup [c, d])$.
2. N is normalized near infinity: $N(z) = \mathbb{I} + \mathcal{O}(1/z)$.
3. Jumps

$$N_+(x) = N_-(x) \begin{pmatrix} 0 & 1/v_1 & 0 \\ -v_1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad x \in (a, b),$$

$$N_+(x) = N_-(x) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1/v_2 \\ 0 & -v_2 & 0 \end{pmatrix}, \qquad x \in (c, d).$$

Global parametrix

We ignore the jumps that tend to $\mathbb{I}$.

The main contribution to S is the matrix N with satisfies the Riemann-Hilbert problem

1. N is analytic in $\mathbb{C} \setminus ([a,b] \cup [c,d])$.
2. N is normalized near infinity: $N(z) = \mathbb{I} + \mathcal{O}(1/z)$.
3. Jumps

$$N_+(x) = N_-(x) \begin{pmatrix} 0 & 1/v_1 & 0 \\ -v_1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad x \in (a,b),$$

$$N_+(x) = N_-(x) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1/v_2 \\ 0 & -v_2 & 0 \end{pmatrix}, \qquad x \in (c,d).$$

4. Convenient behavior near a, b, c, d.

Szegő functions

We write this global parametrix as

$$N(z) = \begin{pmatrix} D_1(\infty) & 0 & 0 \\ 0 & D_2(\infty) & 0 \\ 0 & 0 & D_3(\infty) \end{pmatrix} N_0(z) \begin{pmatrix} D_1(z) & 0 & 0 \\ 0 & D_2(z) & 0 \\ 0 & 0 & D_3(z) \end{pmatrix}^{-1}$$

where (D_1, D_2, D_3) correspond to the Szegő functions of (v_1, v_2) for the Riemann surface $\mathfrak{R}$:

Szegő functions

We write this global parametrix as

$$N(z) = \begin{pmatrix} D_1(\infty) & 0 & 0 \\ 0 & D_2(\infty) & 0 \\ 0 & 0 & D_3(\infty) \end{pmatrix} N_0(z) \begin{pmatrix} D_1(z) & 0 & 0 \\ 0 & D_2(z) & 0 \\ 0 & 0 & D_3(z) \end{pmatrix}^{-1}$$

where (D_1, D_2, D_3) correspond to the Szegő functions of (v_1, v_2) for the Riemann surface $\mathfrak{R}$:

D_1, D_2, D_3 are analytic in $\mathbb{C} \setminus ([a,b] \cup [c,d])$, with $D_k(\infty) \neq 0$ and

$$\begin{cases} D_2^+(x) = v_1(x) D_1^-(x), \\ D_2^-(x) = v_1(x) D_1^+(x), \\ D_3^+(x) = D_3^-(x), \end{cases} , \qquad x \in [a,b],$$

$$\begin{cases} D_1^+(x) = D_1^-(x), \\ D_3^+(x) = v_2(x) D_2^-(x), \\ D_3^-(x) = v_2(x) D_2^+(x), \end{cases} , \qquad x \in [c,d].$$

Standardized global parametrix

The remaining matrix N_0 satisfies the Riemann-Hilbert problem

1. N_0 is analytic in $\mathbb{C} \setminus ([a, b] \cup [c, d])$.

Standardized global parametrix

The remaining matrix N_0 satisfies the Riemann-Hilbert problem

1. N_0 is analytic in $\mathbb{C} \setminus ([a, b] \cup [c, d])$.
2. N_0 is normalized near infinity: $N_0(z) = \mathbb{I} + \mathcal{O}(1/z)$.

Standardized global parametrix

The remaining matrix N_0 satisfies the Riemann-Hilbert problem

1. N_0 is analytic in $\mathbb{C} \setminus ([a,b] \cup [c,d])$.
2. N_0 is normalized near infinity: $N_0(z) = \mathbb{I} + \mathcal{O}(1/z)$.
3. Jumps

$$N_0^+(x) = N_0^-(x) \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad x \in (a,b),$$

$$N_0^+(x) = N_0^-(x) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad x \in (c,d).$$

Standardized global parametrix

The remaining matrix N_0 satisfies the Riemann-Hilbert problem

1. N_0 is analytic in $\mathbb{C} \setminus ([a,b] \cup [c,d])$.
2. N_0 is normalized near infinity: $N_0(z) = \mathbb{I} + \mathcal{O}(1/z)$.
3. Jumps

$$N_0^+(x) = N_0^-(x) \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad x \in (a,b),$$

$$N_0^+(x) = N_0^-(x) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad x \in (c,d).$$

4. Convenient behavior near a, b, c, d.

Final transformation?

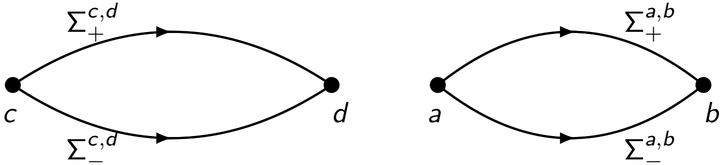
We now consider the matrix

$$R = SN^{-1}.$$

Final transformation?

We now consider the matrix

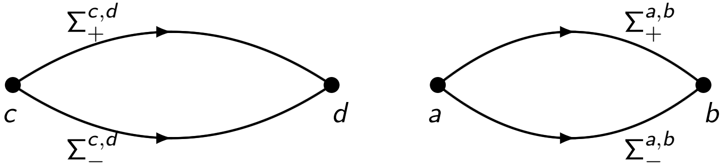
$$R = SN^{-1}.$$



Final transformation?

We now consider the matrix

$$R = SN^{-1}.$$



The jumps on these curves converge to the identity matrix $\mathbb{I}$.

Perturbation result for Riemann-Hilbert problems

Theorem

Suppose R_n satisfies a normalized Riemann-Hilbert problem on a system $\Sigma = \bigcup_{i=1}^{k} \Sigma_i$ of contours. Suppose that the jumps

$$R_n^+(x) = R_n^-(x) J_i \qquad x \in \Sigma_i,$$

converge **uniformly** *to the identity matrix*

$$\lim_{n\to\infty} \|J_i - \mathbb{I}\|_\infty = 0.$$

Then R_n converges uniformly to the identity matrix

$$\lim_{n\to\infty} \|R_n(z) - \mathbb{I}\|_\infty = 0.$$

Final asymptotic result?

$$\lim_{n\to\infty} SN^{-1} = \mathbb{I} \quad \Rightarrow \lim_{n\to\infty} S = N.$$

Final asymptotic result?

$$\lim_{n\to\infty} SN^{-1} = \mathbb{I} \quad \Rightarrow \lim_{n\to\infty} S = N.$$

For z on compact sets of $\mathbb{C} \setminus ([a, b] \cup [c, d])$: outside the lenses $S = V$

$$\lim_{n\to\infty} V = N.$$

Final asymptotic result?

$$\lim_{n\to\infty} SN^{-1} = \mathbb{I} \quad \Rightarrow \lim_{n\to\infty} S = N.$$

For z on compact sets of $\mathbb{C} \setminus ([a,b] \cup [c,d])$: outside the lenses $S = V$

$$\lim_{n\to\infty} V = N.$$

For z on $[a,b]$ or $[c,d]$: inside the lenses $S_+ = V_+ J_+$

$$\lim_{n\to\infty} V_+ J_+ = N_+$$

Final asymptotic result?

$$\lim_{n\to\infty} SN^{-1} = \mathbb{I} \quad \Rightarrow \lim_{n\to\infty} S = N.$$

For z on compact sets of $\mathbb{C} \setminus ([a,b] \cup [c,d])$: outside the lenses $S = V$

$$\lim_{n\to\infty} V = N.$$

For z on $[a,b]$ or $[c,d]$: inside the lenses $S_+ = V_+ J_+$

$$\lim_{n\to\infty} V_+ J_+ = N_+$$

There is a **problem**: the jumps for SN^{-1} on $\Sigma^{a,b}$ and $\Sigma^{c,d}$ do not converge uniformly to $\mathbb{I}$. Uniformity is lost near a, b, c, d.

Final asymptotic result?

$$\lim_{n\to\infty} SN^{-1} = \mathbb{I} \quad \Rightarrow \lim_{n\to\infty} S = N.$$

For z on compact sets of $\mathbb{C} \setminus ([a,b] \cup [c,d])$: outside the lenses $S = V$

$$\lim_{n\to\infty} V = N.$$

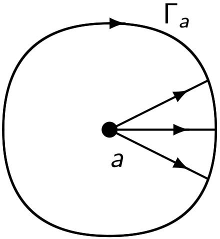
For z on $[a,b]$ or $[c,d]$: inside the lenses $S_+ = V_+ J_+$

$$\lim_{n\to\infty} V_+ J_+ = N_+$$

There is a **problem**: the jumps for SN^{-1} on $\Sigma^{a,b}$ and $\Sigma^{c,d}$ do not converge uniformly to $\mathbb{I}$. Uniformity is lost near a,b,c,d.
A local analysis around a,b,c,d is needed.

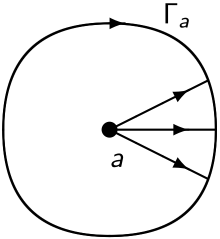
Local parametrices

In the neighborhood of a the Riemann-Hilbert problem for S looks like



Local parametrices

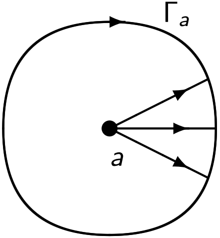
In the neighborhood of a the Riemann-Hilbert problem for S looks like



We approximate it by a model RHP P_a which matches the global parametrix N on the boundary Γ_a with an error $\mathcal{O}(1/n)$.

Local parametrices

In the neighborhood of a the Riemann-Hilbert problem for S looks like

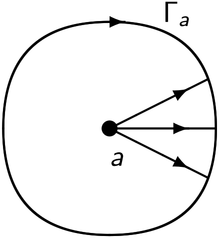


We approximate it by a model RHP P_a which matches the global parametrix N on the boundary Γ_a with an error $\mathcal{O}(1/n)$.

This model Riemann-Hilbert problem uses the **Bessel function** J_α.

Local parametrices

In the neighborhood of a the Riemann-Hilbert problem for S looks like



We approximate it by a model RHP P_a which matches the global parametrix N on the boundary Γ_a with an error $\mathcal{O}(1/n)$.

This model Riemann-Hilbert problem uses the **Bessel function** J_α.

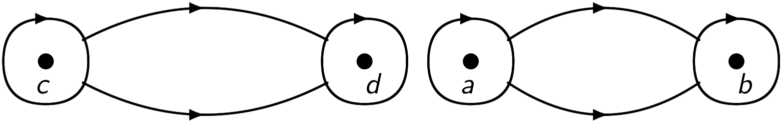
- Around b we need the Bessel function J_β.
- Around c we need the Bessel function J_γ.
- Around d we need the Bessel function J_δ.

The final transformation!

$$R(z) = \begin{cases} SN^{-1}, & z \text{ outside } \Gamma_a, \Gamma_b, \Gamma_c, \Gamma_d, \\ SP_e^{-1}, & z \text{ inside } \Gamma_e,\ e \in \{a, b, c, d\}. \end{cases}$$

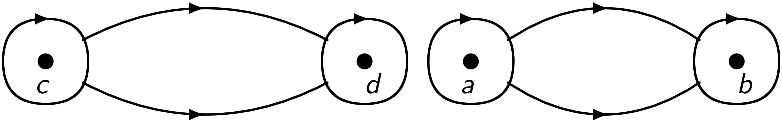
The final transformation!

$$R(z) = \begin{cases} SN^{-1}, & z \text{ outside } \Gamma_a, \Gamma_b, \Gamma_c, \Gamma_d, \\ SP_e^{-1}, & z \text{ inside } \Gamma_e,\ e \in \{a, b, c, d\}. \end{cases}$$



The final transformation!

$$R(z) = \begin{cases} SN^{-1}, & z \text{ outside } \Gamma_a, \Gamma_b, \Gamma_c, \Gamma_d, \\ SP_e^{-1}, & z \text{ inside } \Gamma_e, \; e \in \{a, b, c, d\}. \end{cases}$$



Then

$$\|R_n - \mathbb{I}\| = \mathcal{O}(1/n).$$

Final asymptotic results

Final asymptotic results

Theorem

Let (n, m) be multi-indices that tend to infinity but for which $m/(n+m) = q_1$ remains constant, with $0 < q_1 \leq 1/2$. Then uniformly on compact subsets of $\mathbb{C} \setminus ([a,b] \cup [c,d])$

$$A_{n,m}(z) = [N_1(\psi_1(z)) + \mathcal{O}(1/n)] \frac{D_0(\infty)}{D_1(z)} e^{(n+m)g_1(z) - mg_2(z) + (n+m)\ell_1}$$
$$- [N_1(\psi_2(z)) + \mathcal{O}(1/n)] \frac{D_0(\infty)}{D_2(z)} e^{mg_2(z) + (n+m)(\ell_1+\ell_2)} \int_c^d \frac{d\sigma(t)}{z-t},$$

and

$$B_{n,m}(z) = [N_1(\psi_2(z)) + \mathcal{O}(1/n)] \frac{D_0(\infty)}{D_2(z)} e^{mg_2(z) + (n+m)(\ell_1+\ell_2)}.$$

Final asymptotic results

Theorem

Furthermore

$$A_{n,m}(z) + B_{n,m}(z) \int_c^d \frac{d\sigma(t)}{z-t}$$
$$= [N_1(\psi_1(z)) + \mathcal{O}(1/n)] \frac{D_0(\infty)}{D_1(z)} e^{(n+m)g_1(z) - mg_2(z) + (n+m)\ell_1}.$$

Final asymptotic results: on (c, d)

Theorem

Uniformly on closed subintervals of (c, d) one has

$$B_{n,m}(x) = -2[N_1(\psi_1^+(x)) + \mathcal{O}(1/n)] \frac{D_0(\infty)}{|D_2^+(x)|} e^{-mU(x;\nu_2)} \times \cos\Big(m\pi\varphi_2(x) - \arg D_2^+(x)\Big).$$

Final asymptotic results: on (a, b)

Theorem

Uniformly on closed subintervals of (a, b) *one has*

$$A_{n,m}(x) + B_{n,m}(x) \int_c^d \frac{d\sigma(t)}{x-t} = 2[N_1(\psi_1^+(x)) + \mathcal{O}(1/n)]$$
$$\times \frac{D_0(\infty)}{|D_1^+(x)|} e^{(n+m)U(x;\nu_1)} \cos\Big((n+m)\varphi_1(x) - \arg D_1^+(x)\Big).$$

Type II multiple orthogonal polynomials

Take the inverse-transpose of the Riemann-Hilbert matrix:

$$X^{-T} = \begin{pmatrix} P_{n,m}(z) & \int_a^b \frac{P_{n,m}(t)w_1(t)}{t-z}\,dt & \int_a^b \frac{P_{n,m}(t)w(t)w_1(t)}{t-z}\,dt \\ -\gamma_1 P_{n-1,m}(z) & * & * \\ \gamma_2 P_{n,m-1}(z) & * & * \end{pmatrix}$$

where

$$\begin{aligned} \frac{1}{\gamma_1} &= \int_a^b t^{n-1} P_{n-1,m}(t) w_1(t)\,dt, \\ \frac{1}{\gamma_2} &= \int_a^b t^{m-1} P_{n,m-1}(t) w(t) w_1(t)\,dt. \end{aligned}$$

Asymptotics for type II polynomials

Theorem

Let (n, m) be multi-indices that tend to infinity but for which $m/(n+m) = q_1$ remains constant, with $0 < q_1 \leq 1/2$. Then uniformly on compact subsets of $\mathbb{C} \setminus [a, b]$

$$P_{n,m}(z) = \frac{D_0(z)}{D_0(\infty)} N_1(\psi_0(z)) e^{(n+m)g_1(z)}.$$

Theorem

Let (n, m) be multi-indices that tend to infinity but for which $m/(n+m) = q_1$ remains constant, with $0 < q_1 \leq 1/2$. Then uniformly on compact subsets of $\mathbb{C} \setminus [a, b]$

$$P_{n,m}(z) = \frac{D_0(z)}{D_0(\infty)} N_1(\psi_0(z)) e^{(n+m)g_1(z)}.$$

For x on closed intervals of (a, b) one has

$$P_{n,m}(x) = 2i \frac{|D_0^+(x)|}{D_0(\infty)} [N_1(\psi_0^+(x)) + \mathcal{O}(1/n)] \times \sin\Big((n+m)\varphi_1(x) + \arg D_0^+(x)\Big).$$

References

- P. Deift, X. Zhou, **A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation**, Ann. of Math. (2) 137 (1993), no. 2, 295–368.
- M.E.H. Ismail, **Classical and Quantum Orthogonal Polynomials in One Variable**, Encyclopedia of Mathematics and its Applications 98, Cambridge University Press, 2005 (paperback edition 2009) ; in particular **Chapter 23**.
- Guillermo López Lagomasino, Walter Van Assche, **Riemann-Hilbert analysis for a Nikishin system**, Math. Sbornik (submitted), `arXiv:1612.07108 [math.CA]`
- W. Van Assche, J.S. Geronimo, A.B.J. Kuijlaars, **Riemann-Hilbert problems for multiple orthogonal polynomials**, in 'Special Functions 2000: Current Perspective and Future Directions', (J. Bustoz, M.E.H. Ismail, S.K. Suslov, eds.), NATO Science Series II. Mathematics, Physics and Chemistry, Vol. 30, Kluwer, Dordrecht, 2001, pp. 23–59.