Properties of orthogonal polynomials

Kerstin Jordaan

University of South Africa

LMS Research School

University of Kent, Canterbury

Outline

1. Orthogonal polynomials
2. Properties of classical orthogonal polynomials
   - Characterizing properties
   - Rodrigues type formulas
   - The Askey and q-Askey scheme
   - Generalised hypergeometric polynomials
   - Zeros of classical orthogonal polynomials
   - Markov's monotonicity theorem
   - A conjecture due to Askey
3. Quasi-orthogonality and semiclassical orthogonal polynomials
4. The hypergeometric function
5. Convergence of Padé approximants for a hypergeometric function

The inverse problem

Operator theory has played an important role in answering questions concerning the

- existence
- uniqueness
- nature
- support

of the orthogonality measure.

The inverse problem

Operator theory has played an important role in answering questions concerning the

- existence
- uniqueness
- nature
- support

of the orthogonality measure.

For some classical references on this topic see

- Stone [1932] for a discussion on Jacobi matrices
- Sarason [1987] for background on the moment problem
- Shohat and Tamarkin [1950] for more technical information on the moment problem
- Kato, [1957] for perturbation theorems
- Berezanskii [1968] for a discussion of self-adjointedness

The direct problem

Given an orthogonality measure μ, several characteristic properties of the sequence $\{P_n\}$ are determined by the nature of the measure.

Extracting this information from the measure is one of the interesting problems in the study of systems of orthogonal polynomials.

Properties typically studied include

- the Hankel determinants
- the coefficients of the three term recurrence relation
- the coefficients of the differential-difference equation
- the coefficients of the differential equation

satisfied by the polynomials, if at all.

The direct problem

Given an orthogonality measure μ, several characteristic properties of the sequence $\{P_n\}$ are determined by the nature of the measure.

Extracting this information from the measure is one of the interesting problems in the study of systems of orthogonal polynomials.

Properties typically studied include

- the Hankel determinants
- the coefficients of the three term recurrence relation
- the coefficients of the differential-difference equation
- the coefficients of the differential equation

satisfied by the polynomials, if at all.

For classical orthogonal polynomials, namely Hermite, Laguerre and Jacobi polynomials, the properties that they satisfy are well known.

Consider the very classical Hermite, Laguerre and Jacobi polynomials.

Consider the very classical Hermite, Laguerre and Jacobi polynomials.

(a) Their derivatives also form orthogonal polynomial sets.

(b) They all satisfy a second order linear differential equation of the Sturm-Liouville type

$$\sigma(x)P_n''(x) + \tau(x)P_n'(x) + \lambda_n P_n(x) = 0 \tag{1}$$

where $\sigma(x)$ is a polynomial of degree ≤ 2, $\tau(x)$ is a linear polynomial, both independant of n, and λ_n is independant of x. Equivalently, the weights satisfy a first-order differential equation, the Pearson equation

$$\frac{d}{dx}[\sigma(x)w(x)] = \tau(x)w(x),$$

with $\sigma(x)$ and $\tau(x)$ the same polynomials as in (1).

Classical orthogonal polynomials all satisfy

(c) a non-linear equation of the form

$$\sigma(x)\frac{d}{dx}P_n(x)P_{n-1}(x) = (\alpha_n\ x + \beta_n)P_n(x)P_{n-1}(x) + \gamma_n P_n^2(x) + \delta_n P_{n-1}^2(x)$$

where $\alpha_n,\ \beta_n,\ \gamma_n$ and δ_n are independent of x.

(d) a differential-difference relation

$$\pi(x)P_n'(x) = (a_n\ x + b_n)P_n(x) + c_n P_{n-1}.$$

(e) a Rodrigues' type formula

$$P_n(x) = \frac{1}{a_n\ w(x)} D^n[w(x)\sigma^n(x)], \qquad n = 0, 1, 2, \ldots$$

where $\sigma(x)$ is a polynomial in x independent of n and a_n does not depend on x.

Classical orthogonal polynomials all satisfy

(c) a non-linear equation of the form

$$\sigma(x)\frac{d}{dx}P_n(x)P_{n-1}(x) = (\alpha_n\ x + \beta_n)P_n(x)P_{n-1}(x) + \gamma_n P_n^2(x) + \delta_n P_{n-1}^2(x)$$

where $\alpha_n,\ \beta_n,\ \gamma_n$ and δ_n are independent of x.

(d) a differential-difference relation

$$\pi(x)P_n'(x) = (a_n\ x + b_n)P_n(x) + c_n P_{n-1}.$$

(e) a Rodrigues' type formula

$$P_n(x) = \frac{1}{a_n\ w(x)} D^n[w(x)\sigma^n(x)], \qquad n = 0, 1, 2, \ldots$$

where $\sigma(x)$ is a polynomial in x independent of n and a_n does not depend on x.

Any polynomial set which satisfies any one of the above properties must necessarily be one of the classical orthogonal polynomial sets.

Rodrigues formula

- The Rodrigues formula provides transparent and immediate information about
  - the interval of orthogonality
  - the weight function, and
  - the range of parameters for which orthogonality holds.

Rodrigues formula

- The Rodrigues formula provides transparent and immediate information about
  - the interval of orthogonality
  - the weight function, and
  - the range of parameters for which orthogonality holds.

  Example: We will use Jacobi's formula and illustrate how one derives an orthogonality relation.

Rodrigues formula

- The Rodrigues formula provides transparent and immediate information about
  - the interval of orthogonality
  - the weight function, and
  - the range of parameters for which orthogonality holds.

  Example: We will use Jacobi's formula and illustrate how one derives an orthogonality relation.
- The Rodrigues formula for the Legendre polynomials
  - published by O. Rodrigues in an Ecole Polytechnique journal [1816].
  - Rodrigues' paper did not receive much attention.
  - Rediscovered independently by J. Ivory [1822] and Jacobi [1827]. Jacobi later suggested to Ivory that they write a joint paper on the result and publish it in France since it was not known there!! Their paper appeared in Liouville's journal in 1837.

Rodrigues formula

- The Rodrigues formula provides transparent and immediate information about
  - the interval of orthogonality
  - the weight function, and
  - the range of parameters for which orthogonality holds.

  Example: We will use Jacobi's formula and illustrate how one derives an orthogonality relation.
- The Rodrigues formula for the Legendre polynomials
  - published by O. Rodrigues in an Ecole Polytechnique journal [1816].
  - Rodrigues' paper did not receive much attention.
  - Rediscovered independently by J. Ivory [1822] and Jacobi [1827]. Jacobi later suggested to Ivory that they write a joint paper on the result and publish it in France since it was not known there!! Their paper appeared in Liouville's journal in 1837.
- Laplace, who was Rodrigues' supervisor found a similar result for the Hermite polynomials in his work on probability in 1810.

What defines a classical orthogonal polynomial?

A sequence of orthogonal polynomials is classical if the sequence $\{P_n\}$ as well as $D^m P_{n+m}$, $m \in \mathbb{N}$, where D is the usual derivative $\frac{d}{dx}$ or one of its extensions

- difference operator
- q-difference operator
- divided-difference operator

satisfies a three term recurrence of the form ensuring orthogonality by the spectral theorem.

What defines a classical orthogonal polynomial?

A sequence of orthogonal polynomials is classical if the sequence $\{P_n\}$ as well as $D^m P_{n+m}$, $m \in \mathbb{N}$, where D is the usual derivative $\frac{d}{dx}$ or one of its extensions

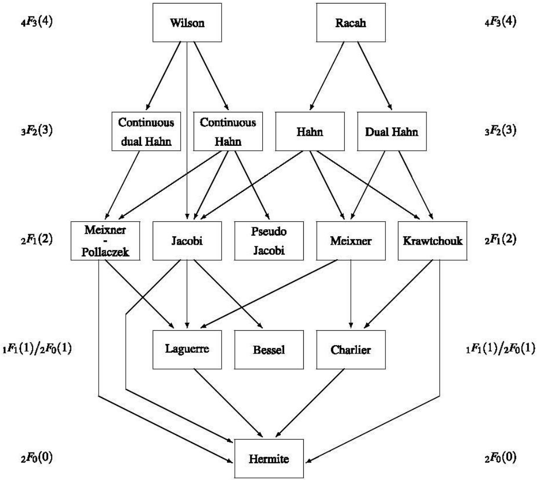
- difference operator
- q-difference operator
- divided-difference operator

satisfies a three term recurrence of the form ensuring orthogonality by the spectral theorem.

The following definition of classical orthogonal polynomials, suggested by Andrews and Askey in 1985 is generally accepted and has been justified by various characterizations.

"A set of orthogonal polynomials is classical, if it is a special case or limiting case of the Askey-Wilson polynomials"

ASKEY SCHEME OF HYPERGEOMETRIC ORTHOGONAL POLYNOMIALS

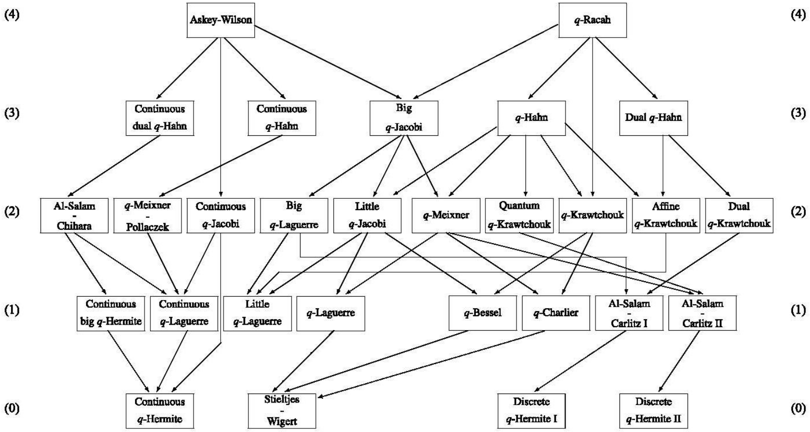


SCHEME

OF

BASIC HYPERGEOMETRIC

ORTHOGONAL POLYNOMIALS



Generalised hypergeometric polynomials

Generalised hypergeometric polynomials

$$ {}_pF_q\left(-n, \alpha_2, \alpha_3, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; x\right) = \sum_{m=0}^{n} \frac{(-n)_m(\alpha_2)_m(\alpha_3)_m \ldots (\alpha_p)_m}{(\beta_1)_m \ldots (\beta_q)_m} \frac{x^m}{m!} $$

where $\beta_1, \ldots, \beta_q \notin \{0, -1, -2, \ldots, \}$ and

$$ (\beta)_k = \beta(\beta+1)\ldots(\beta+k-1) $$

is Pochhammer's symbol, also known as the shifted factorial.

Generalised hypergeometric polynomials

$$ {}_pF_q\left(-n, \alpha_2, \alpha_3, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; x\right) = \sum_{m=0}^{n} \frac{(-n)_m(\alpha_2)_m(\alpha_3)_m \ldots (\alpha_p)_m}{(\beta_1)_m \ldots (\beta_q)_m} \frac{x^m}{m!} $$

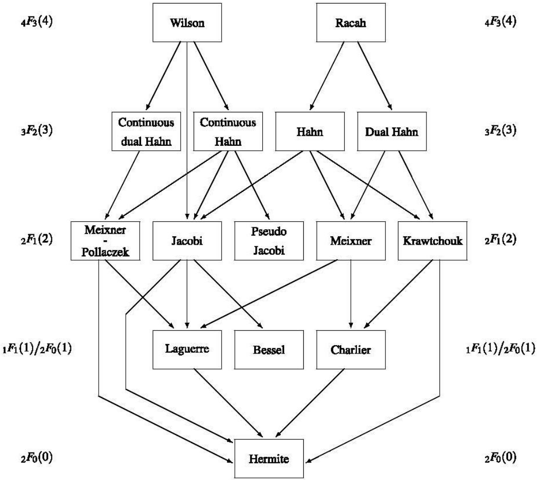
where $\beta_1, \ldots, \beta_q \notin \{0, -1, -2, \ldots,\}$ and

$$ (\beta)_k = \beta(\beta+1)\ldots(\beta+k-1) $$

is Pochhammer's symbol, also known as the shifted factorial.

Not all ${}_pF_q$ polynomials are orthogonal.

ASKEY SCHEME OF HYPERGEOMETRIC ORTHOGONAL POLYNOMIALS



Zeros of classical orthogonal polynomials

Theorem

If $\{p_n(x)\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomials on the interval (a, b) with respect to the weight function $w(x)$, then the polynomial $p_n(x)$ has exactly n real simple zeros in the interval (a, b).

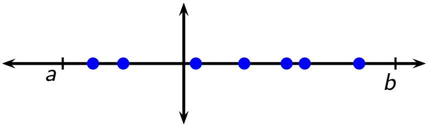


Figure: Zeros in interval I

Proof

Since $\deg(p_n) = n$ the polynomial has at most n real zeros.

Suppose that $p_n(x)$ has $m \leq n$ distinct real zeros $x_1, x_2, \ldots, x_m$ in (a, b) of odd order (or multiplicity).

Then the polynomial

$$p_n(x)(x - x_1)(x - x_2) \ldots (x - x_m)$$

does not change sign on the interval (a, b).

This implies that

$$\int_a^b w(x) p_n(x)(x - x_1)(x - x_2) \ldots (x - x_m) dx \neq 0.$$

By orthogonality this integral equals zero if $m < n$.

Hence: $m = n$, which implies that $p_n(x)$ has n distinct real zeros of odd order in (a, b).

This proves that all n zeros are distinct and simple (have order or multiplicity equal to one).

An important consequence of the recurrence relation, is the Christoffel-Darboux formula (See Assignment 1, Ex. 3).

Theorem

A sequence of orthogonal polynomials $\{p_n(x)\}_{n=0}^{\infty}$ *satisfies*

$$\sum_{k=0}^{n} \frac{p_k(x)p_k(y)}{h_k} = \frac{k_n}{h_n\, k_{n+1}} \frac{p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)}{x-y}, \; n = 0,\, 1, 2, \ldots \quad (2)$$

and

$$\sum_{k=0}^{n} \frac{\{p_k(x)\}^2}{h_k} = \frac{k_n}{h_n\, k_{n+1}} (p'_{n+1}(x)p_n(x) - p_{n+1}(x)p'_n(x)), \; n = 0,\, 1, 2, \ldots \quad (3)$$

(2) is called the Christoffel-Darboux formula and (3) its confluent form.

(3) yields another important property of zeros of orthogonal polynomials.

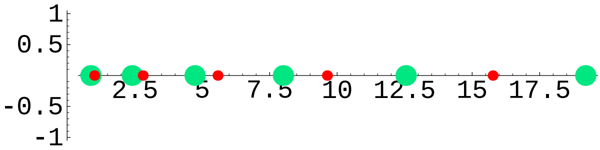
Interlacing of zeros

Theorem

If $\{p_n(x)\}_{n=0}^{\infty}$ is a sequence of orthogonal polynomial on the interval (a, b) with respect to the weight function $w(x)$, then the zeros of $p_n(x)$ and $p_{n+1}(x)$ separate each other.

f $\{x_{n,\,k}\}_{k=1}^{n}$ and $\{x_{n+1,\,k}\}_{k=1}^{n+1}$ denote the consecutive zeros of $p_n(x)$ and $p_{n+1}(x)$ respectively, then we have

$$a < x_{n+1,\,1} < x_{n,\,1} < x_{n+1,\,2} < x_{n,\,2} < \ldots < x_{n+1,\,n} < x_{n,\,n} < x_{n+1,\,n+1} < b.$$



Proof

Note that

$$h_n = \int_a^b w(x) \{p_n(x)\}^2 \; dx > 0, \;\; n = 0, 1, 2, \ldots$$

This implies that

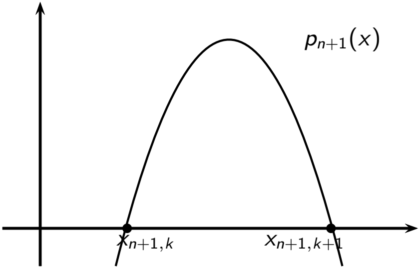
$$\frac{k_n}{h_n \, k_{n+1}} (p'_{n+1}(x) p_n(x) - p_{n+1}(x) p'_n(x)) = \sum_{k=0}^{n} \frac{\{p_k(x)\}^2}{h_k} > 0$$

Hence

$$\frac{k_n}{k_{n+1}} (p'_{n+1}(x) p_n(x) - p_{n+1}(x) p'_n(x)) > 0.$$

Now suppose that $x_{n+1,k}$ and $x_{n+1,k+1}$, $k = 1, 2, \ldots, n$ are any two consecutive zeros of $p_{n+1}(x)$ with $x_{n+1,k} < x_{n+1,k+1}$.

Since all $n + 1$ zeros of $p_{n+1}(x)$ are real and simple, it follows from Rolle's theorem that $p'_{n+1}(x_{n+1,k})$ and $p'_{n+1}(x_{n+1,k+1})$ have opposite signs.

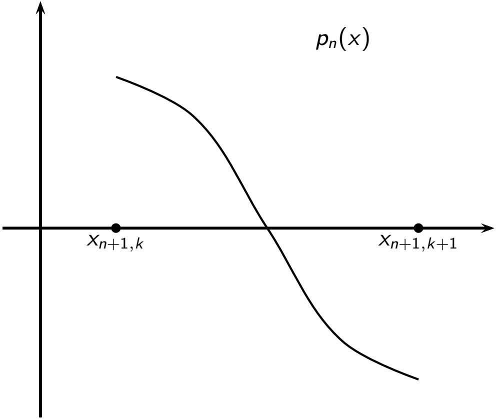


Hence we have

$$p_{n+1}(x_{n+1,\,k}) = 0 = p_{n+1}(x_{n+1,k+1}) \text{ and } p'_{n+1}(x_{n+1,k})p'_{n+1}(x_{n+1,k+1}) < 0.$$

This implies that $p_n(x_{n+1,\,k})p_n(x_{n+1,\,k+1}) < 0$. Why?

It follows from the continuity of $p_n(x)$ that there should be at least one zero of $p_n(x)$ between $x_{n+1,\,k}$ and $x_{n+1,\,k+1}$.



This holds for each pair of consecutive zeros of $p_{n+1}(x)$ so there is exactly one zero of $p_n(x)$ between $x_{n+1,\,k}$ and $x_{n+1,\,k+1}$.

Monotonicity of the zeros

The manner in which the zeros of a polynomial change as the parameter changes can be used to study interlacing properties of zeros.

Markov's monotonicity theorem [1886] proves that the zeros of classical orthogonal polynomials like Laguerre and Jacobi polynomials are monotone functions of the parameter(s) by using the derivative of the weight function with respect to the parameter(s).

A slightly generalised version of Markov's theorem, stated as an exercise in Freud's book [1971] and proved in Ismail's book [2005] can also be applied to discrete orthogonal polynomials such as Meixner and Hahn polynomials.

A monotonicity theorem due to Markov

Theorem

Let $\{p_n(x,\tau)\}_{n=0}^{\infty}$ be orthogonal with respect to $d\alpha(x,\tau) = w(x,\tau)d\alpha(x)$ on the interval $[a,b]$ depending on a parameter τ such that $w(x,\tau)$ is positive and continuous for $a < x < b$, $\tau_1 < \tau < \tau_2$.

Also, suppose that the partial derivative $w_\tau(x,\tau)$ for $a < x < b$, $\tau_1 < \tau < \tau_2$ exists and is continuous, and the integrals

$$\int_a^b x^\nu w_\tau(x,\tau)d\alpha(x),\ \nu = 0,1,2,\ldots,2n-1,$$

converge uniformly in every closed interval $[\tau', \tau''] \subset (\tau_1, \tau_2)$.

Denote the zeros of $p_n(x,\tau)$ by

$$x_1(\tau) > x_2(\tau) > \cdots > x_n(\tau) > 0.$$

Then the νth zero $x_\nu(\tau)$ (for a fixed value of ν) is an increasing (decreasing) function of τ provided that

$$w_\tau/w$$

is an increasing (decreasing) function of x, $a < x < b$.

For α, $\beta > -1$, the sequence of Jacobi polynomials $\{P_n^{\alpha,\beta}\}_{n=0}^{\infty}$ is orthogonal with respect to the weight function $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$ on $(-1,1)$ and satisfies the three term recurrence relation

$$\begin{aligned} &\frac{2(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} P_{n+1}^{\alpha,\beta}(x) \\ = \quad &\left(x - \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}\right) P_n^{\alpha,\beta}(x) \\ &- \frac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)} P_{n-1}^{\alpha,\beta}(x). \end{aligned}$$

Example

For Jacobi polynomials $P_n(\alpha,\beta)$, the weight function is $w(x,\alpha,\beta) = (1-x)^{\alpha}(1+x)^{\beta}$ and $\alpha(x) = x$, hence

$$\frac{\partial \ln w(x,\alpha,\beta)}{\partial\beta} = \frac{\partial \ln(1+x)^{\beta}}{\partial\beta} = \ln(1+x)$$

which is an increasing function of x.

The variation of the zeros of a Jacobi polynomial with the parameter can be summarised as follows.

Lemma (cf. (Szego, Theorem 6.21.1, p.121))

Let $\alpha > -1$ and $\beta > -1$ and let x_k, $k = 1, 2, \ldots, n$ denote the zeros of $P_n^{(\alpha,\beta)}$ in increasing order. Then $\dfrac{dx_k}{d\alpha} < 0$ and $\dfrac{dx_k}{d\beta} > 0$ for each $k = 1, \ldots, n$.

Richard Askey [1990] in "graphs as an aid to understanding special functions" conjectured the following.

Conjecture

The zeros of the Jacobi polynomials $P_n^{(\alpha,\beta)}$ and $P_n^{(\alpha,\beta+2)}$ are interlacing for each $n \in \mathbb{N}$, $\alpha,\ \beta > -1$.

Interlacing of zeros of different Jacobi polynomials

Interlacing of zeros of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ and $P_{n\pm\epsilon}^{(\alpha\pm t,\beta\pm k)}(x)$.

Interlacing of zeros of different Jacobi polynomials

Interlacing of zeros of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ and $P_{n\pm\epsilon}^{(\alpha\pm t,\beta\pm k)}(x)$.

- Must t and k be integers?

Interlacing of zeros of different Jacobi polynomials

Interlacing of zeros of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ and $P_{n\pm\epsilon}^{(\alpha\pm t,\beta\pm k)}(x)$.

- Must t and k be integers? NO.

Interlacing of zeros of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ and $P_{n\pm\epsilon}^{(\alpha\pm t,\beta\pm k)}(x)$.

- Must t and k be integers? NO.

  $\epsilon = 0$ or 1 and $0 < t, k \leq 2$ (Driver, J and Mbuyi [2008])

Interlacing of zeros of different Jacobi polynomials

Interlacing of zeros of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ and $P_{n\pm\epsilon}^{(\alpha\pm t,\beta\pm k)}(x)$.

- Must t and k be integers? NO.

  $\epsilon = 0$ or 1 and $0 < t, k \leq 2$ (Driver, J and Mbuyi [2008])

- Must ϵ be an integer?

Interlacing of zeros of different Jacobi polynomials

Interlacing of zeros of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ and $P_{n\pm\epsilon}^{(\alpha\pm t,\beta\pm k)}(x)$.

- Must t and k be integers? NO.

  $\epsilon = 0$ or 1 and $0 < t, k \leq 2$ (Driver, J and Mbuyi [2008])

- Must ϵ be an integer? NO.

Interlacing of zeros of different Jacobi polynomials

Interlacing of zeros of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ and $P_{n\pm\epsilon}^{(\alpha\pm t,\beta\pm k)}(x)$.

- Must t and k be integers? NO.

  $\epsilon = 0$ or 1 and $0 < t, k \leq 2$ (Driver, J and Mbuyi [2008])

- Must ϵ be an integer? NO.

  $0 < \epsilon < 1$ and $t, k = 0, 1, 2$ (Segura, [2008])

Theorem

Let $\alpha > 1$, $\beta > -1$ *and* $t \in (0,2)$, $k \in (0,2)$.

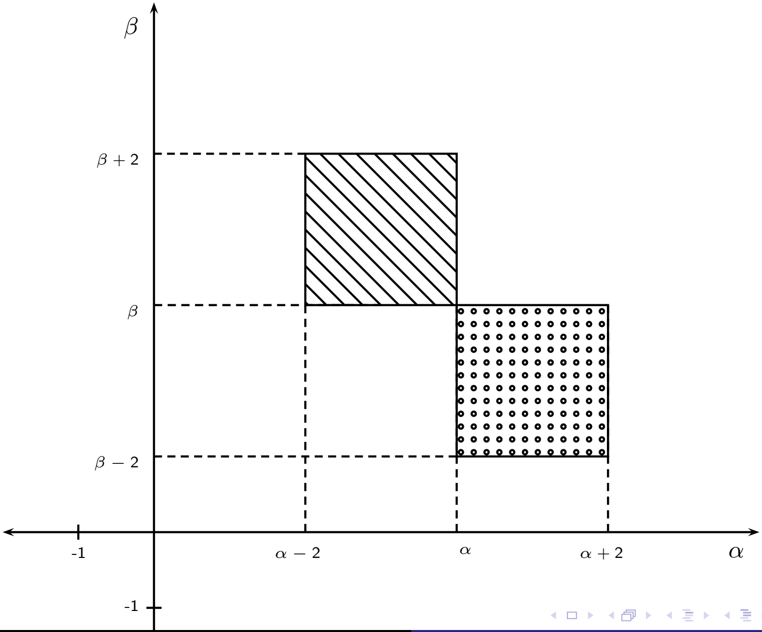
$$\begin{array}{llll} & \textit{Let} -1 < x_1 < x_2 < \cdots < x_n < 1, & \textit{be the zeros of} & P_n^{(\alpha,\beta)}, \\ & -1 < t_1 < t_2 < \cdots < t_n < 1, & \textit{be the zeros of} & P_n^{(\alpha-k,\beta+t)}, \\ \textit{and} & -1 < y_1 < y_2 < \cdots < y_n < 1, & \textit{be those of} & P_n^{(\alpha-2,\beta+2)}. \end{array}$$

Then

$$-1 < x_1 < t_1 < y_1 < x_2 < t_2 < y_2 < \cdots < x_n < t_n < y_n < 1.$$

It follows by symmetry that the zeros of $P_n^{(\alpha,\beta)}$ and $P_n^{(\alpha-t,\beta-k)}$, $t, k > 0$ also do not interlace in general.

Interlacing of zeros of $P_n^{(\alpha,\beta)}(x)$ and $P_n^{(\alpha',\beta')}(x)$, $\alpha, \alpha', \beta, \beta' > -1$



Theorem

Let $\alpha,\ \beta > -1$ *and let* $0 \leq t \leq 2$ *and* $0 \leq k \leq 2$*. Let*

$$\begin{aligned} -1 < x_1 < x_2 < \cdots < x_n < 1 &\quad \text{be the zeros of} \quad P_n^{(\alpha,\beta)} \text{ and} \\ -1 < t_1 < t_2 < \cdots < t_{n-1} < 1 &\quad \text{be the zeros of} \quad P_{n-1}^{(\alpha+t,\beta+k)}. \end{aligned}$$

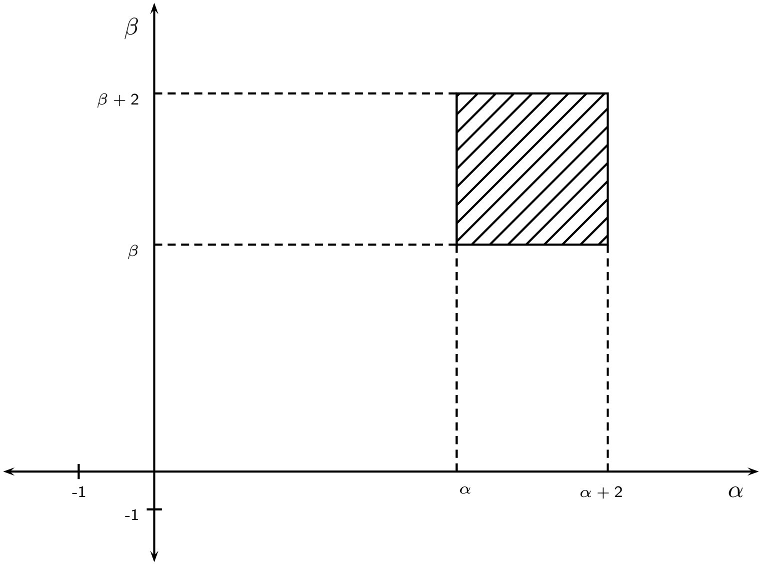
Then

$$-1 < x_1 < t_1 < x_2 < \cdots < x_{n-1} < t_{n-1} < x_n < 1.$$

Remark

Some restrictions on the ranges of t *and* k *are required in the theorems since the interlacing property is not retained, in general, when one or both of the parameters* $\alpha,\ \beta$ *are increased by more than* 2*.*

Interlacing of zeros of $P_n^{(\alpha,\beta)}$ and $P_{n-1}^{(\alpha',\beta')}$ $\alpha, \beta > -1$



Some final comments

The proofs make extensive use of

- the Markov monotonicity theorem as applied to Jacobi polynomials
- the contiguous relations for ${}_2F_1$ hypergeometric polynomials. Various algorithms have been developed for computing such contiguous relations.

Dimitrov, Ismail and Rafaeli [2013] used a general approach to the Askey Conjecture by considering interlacing properties of zeros of orthogonal Jacobi polynomials $P_n^{(\alpha,\beta)}$ of the same degree and different parameter values α and β in the context of perturbation of the weight function of orthogonality.

Proof of the monotonicity theorem

The mechanical quadrature formula (see, for example, Ismail's book, (2.4.1))

$$\int_a^b \rho(x)d\alpha(x,\tau) = \sum_{\nu=1}^{n} \lambda_\nu(\tau)\rho(x_\nu(\tau)), \tag{4}$$

holds for polynomials $\rho(x)$ of degree at most $2n-1$. Differentiating (4) with respect to τ, we obtain

$$\int_a^b \rho(x)w_\tau(x,\tau)d\alpha(x) = \sum_{\nu=1}^{n} \lambda_\nu(\tau)\rho'(x_\nu)x_\nu'(\tau) + \sum_{\nu=1}^{n} \lambda_\nu'(\tau)\rho(x_\nu).$$

Now we choose

$$\rho(x) = \frac{\{p_n(x,\tau)\}^2}{x-x_\nu},$$

then, since x_ν is a removable singularity, $\rho'(x_\nu) = \{p_n'(x_\nu,\tau)\}^2$ while $\rho'(x_\mu) = 0$ if $\mu \neq \nu$ and hence

$$\int_a^b w_\tau(x,\tau)\frac{\{p_n(x,\tau)\}^2}{x-x_\nu}d\alpha(x) = \lambda_\nu(\tau)\{p_n'(x_\nu,\tau)\}^2 x_\nu'(\tau). \tag{5}$$

In view of the orthogonality the integral

$$\int_a^b \frac{\{p_n(x,\tau)\}^2}{x-x_\nu} w(x,\tau)\,d\alpha(x) = 0,$$

so (5) can be rewritten as

$$\int_a^b \left\{ w_\tau(x,\tau) - \frac{w_\tau(x_\nu,\tau)}{w(x_\nu,\tau)} w(x,\tau) \right\} \frac{\{p_n(x,\tau)\}^2}{x-x_\nu} d\alpha(x) = \lambda_\nu(\tau)\{p_n^{'}(x_\nu,\tau)\}^2 x_\nu^{'}(\tau).$$

and we obtain

$$\int_a^b \left\{ \frac{w_\tau(x,\tau)}{w(x,\tau)} - \frac{w_\tau(x_\nu,\tau)}{w(x_\nu,\tau)} \right\} \frac{\{p_n(x,\tau)\}^2}{x-x_\nu} d\alpha(x,\tau) = \lambda_\nu(\tau)\{p_n^{'}(x_\nu,\tau)\}^2 x_\nu^{'}(\tau). \tag{6}$$

The integrand in (6) has a constant sign, so the positivity of the so-called Christoffel numbers $\lambda_\nu(\tau)$ (cf. [Szegö, p. 48] establishes the result.