Properties of orthogonal polynomials

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Outline

Quasi-orthogonality

Definition

A polynomial R_n, $\deg R_n = n$, $n \geq r$ is quasi-orthogonal of order r where $n, r \in \mathbb{N}$ with respect to $w(x) > 0$ on I if

$$\int_I x^k R_n(x) w(x)\, dx \begin{cases} = 0 & \text{for} \quad k = 0, 1, \ldots, n-r-1 \\ \neq 0 & \text{for} \quad k = n-r. \end{cases}$$

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A characterisation of quasi-orthogonality of order r:

Theorem (Shohat)

Let $\{P_n\}_{n=0}^{\infty}$ be a family of orthogonal polynomials with respect to $w(x) > 0$ on $[a, b]$. Then the n-th degree polynomial R_n is quasi-orthogonal of order r on $[a, b]$ with respect to $w(x)$ if and only if there exist constants c_i, $i = 0, \ldots, r$ and $c_0 c_r \neq 0$ such that

$$R_n(x) = c_0 P_n(x) + c_1 P_{n-1}(x) + \ldots + c_r P_{n-r}(x).$$

Historical overview

- Riesz [1923]: Quasi-orthogonal polynomials of order 1

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Marcel Riesz

Historical overview

- Riesz [1923]: Quasi-orthogonal polynomials of order 1

Figure: Marcel and Frigyes Riesz

Historical overview

Figure: Lipót Fejér

- Fejér [1933]: Quasi-orthogonality of order 2
- Shohat [1937]: Quasi-orthogonality of any order r

More recently

- Chihara [1957]: Generalised definition of quasi-orthogonal polynomials and studied them in the context of three-term recurrence relations
- Dickinson [1961]: System of recurrence relations necessary and sufficient for quasi-orthogonality of order 1
- Draux [1990]: Proved the converse of one of Chihara's results
- Brezinski, Driver, Redivo-Zaglia [2004]: Results on the real zeros of quasi-orthogonal polynomials
- Joulak [2005]: Extended these results by giving necessary and sufficient conditions

A result by Dickinson

Dickinson applied systems of recurrence relations that are necessary and sufficient for quasi-orthogonality to some special cases of Fasenmyer polynomials

$$f_n(a,x) = {}_3F_2\left(\begin{array}{c} -n, n+1, a \\ \frac{1}{2}, 1 \end{array}; x\right) = \sum_{m=0}^{n} \frac{(-n)_m (n+1)_m (a)_m}{\left(\frac{1}{2}\right)_m (1)_m} \frac{x^m}{m!}.$$

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Theorem (Dickenson, 1961)

The polynomials $f_n\left(\frac{3}{2}, x\right)$ and $f_n(2,x)$ are quasi-orthogonal of order 1 on the interval $(0,1)$ with weights $(1-x)$ and $x^{-1/2}(1-x)^{3/2}$ respectively.

These turn out to be very special cases of more general classes of quasi-orthogonal ${}_pF_q$ polynomials arising from orthogonal ${}_{p-1}F_{q-1}$ polynomials (cf. Johnston and J [2015]).

Sister Celine

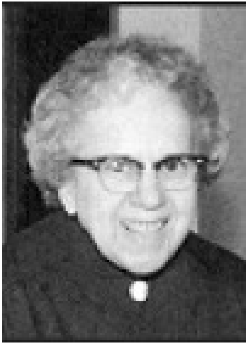

Figure: Celine Fasenmyer

Zeros of quasi-orthogonal polynomials of order r

Theorem (Shohat)

If R_n is quasi-orthogonal of order r on $[a, b]$ with respect to a positive weight function, then at least $(n - r)$ distinct zeros of R_n lie in the interval (a, b)

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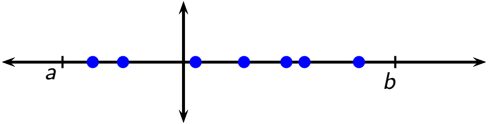

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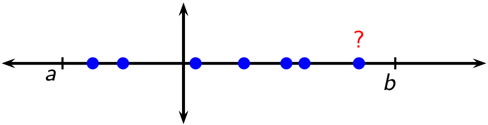

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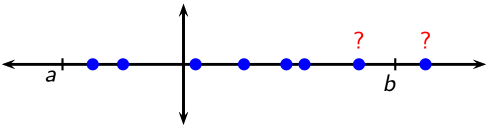


Figure: Quasi-orthogonality of order 1: at least $n - 1$ zeros in interval I

Quasi-orthogonal polynomials of order 2

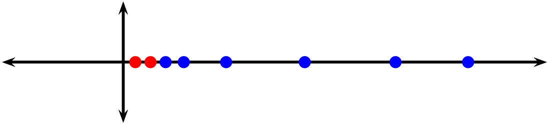
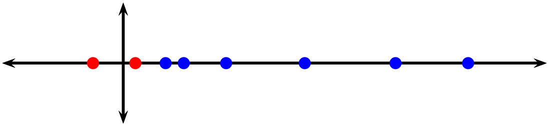


Figure: Quasi-orthogonality of order 2: at least $n-2$ zeros in interval I

Quasi-orthogonal polynomials of order 2



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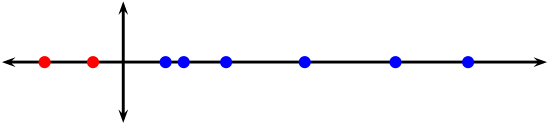

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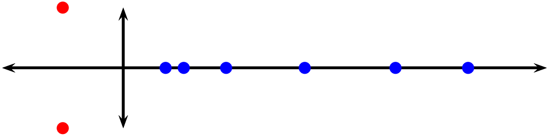

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Semiclassical orthogonal polynomials

Al-Salam and Chihara [1972] showed that orthogonal polynomial sets satisfying

$$\pi(x)P_n'(x) = (a_n\ x + b_n)P_n(x) + c_nP_{n-1}$$

must be either Hermite, Laguerre or Jacobi polynomials.

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Askey raised the more general question of what orthogonal polynomial sets have the property that their derivatives satisfy a relation of the form

$$\pi(x)P_n'(x) = \sum_{k=n-t}^{n+s} \alpha_{nk}P_k(x).$$

This problem was considered by Shohat [1939] and later, independently, by Freud [1976], as well as Bonan and Nevai [1984].

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Maroni [1985] stated the problem in a different way, trying to find all orthogonal polynomial sets whose derivatives are quasi-orthogonal, and called such orthogonal polynomial sets semi-classical.

Semiclassical orthogonal polynomials

Consider the Pearson equation satisfied by the weight $w(x)$

$$\frac{d}{dx}[\sigma(x)w(x)] = \tau(x)w(x)$$

Classical orthogonal polynomials: $\sigma(x)$ and $\tau(x)$ are polynomials with $\deg(\sigma) \leq 2$ and $\deg(\tau) = 1$

	$w(x)$	$\sigma(x)$	$\tau(x)$
Hermite	$exp(-x^2)$	1	$-2x$
Laguerre	$x^\alpha exp(-x)$	x	$1+\alpha-x$
Jacobi	$(1-x)^\alpha(1+x)^\beta$	$(1-x)^2$	$\beta-\alpha-(2+\alpha+\beta)x$

Semi-classical orthogonal polynomials: $\sigma(x)$ and $\tau(x)$ are polynomials with either $\deg(\sigma) > 2$ or $\deg(\tau) > 1$

	$w(x)$	$\sigma(x)$	$\tau(x)$
semi-classical Laguerre	$x^\lambda exp(-x^2+tx)$	x	$1+\lambda+tx-2x^2$
generalized Freud	$\lvert x\rvert^{2\lambda+1} exp(-x^4+2tx^2)$	x	$2\lambda+2-2tx^2-x^4$

Extract from *Digital Library of Mathematical Functions*

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18.32 OP's with Respect to Freud Weights

A *Freud weight* is a weight function of the form

$$w(x) = exp(-Q(x)), \quad -\infty < x < \infty$$

where $Q(x)$ is real, even, nonnegative, and continuously differentiable. Of special interest are the cases $Q(x) = x^{2m}$, $m = 1, 2, \ldots$. No explicit expressions for the corresponding OP's are available. However, for asymptotic approximations in terms of elementary functions for the OP's, and also for their largest zeros , see Levin and Lubinsky [2001] and Nevai [1986]. For a uniform asymptotic expansion in terms of Airy functions for the OP's in the case x^4 see Bo and Wong [1999].

The generalized Freud weight

Expressions for the recurrence coefficients associated with the semi-classical Laguerre weight

$$w(x) = x^{\lambda} \exp(-x^2 + tx), \quad x \in (0, \infty) \ \text{for } \lambda > -1 \text{ and } t \in \mathbb{R}$$

and the generalized Freud weight

$$w(x) = |x|^{2\lambda+1} \exp(-x^4 + tx^2) \quad x \in \mathbb{R} \ \text{for } \lambda > -1 \text{ and } t \in \mathbb{R}$$

can be given in terms of Wronskians of parabolic cylinder functions that appear in the description of special function solutions of the fourth Painlevé equation and discrete Painlevé1.

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The link between the theory of Painlevé equations and orthogonal polynomials is given by the moments of the weight

$$\mu_n = \int_a^b x^n w(x)\, dx, \quad n = 0, 1, 2, \ldots$$

which allow the Hankel determinant to be written as a Wronskian.

Moments of the generalized Freud weight

The first moment, $\mu_0(t;\lambda)$, can be obtained using the integral representation of a parabolic cylinder function.

$$\mu_0(t;\lambda) = \int_{-\infty}^{\infty} |x|^{2\lambda+1} \exp\left(-x^4 + tx^2\right) dx$$

$$= \frac{\Gamma(\lambda+1)}{2^{(\lambda+1)/2}} \exp\left(\tfrac{1}{8}t^2\right) D_{-\lambda-1}\left(-\tfrac{1}{2}\sqrt{2}\,t\right).$$

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The even moments are

$$\begin{aligned}\mu_{2n}(t;\lambda) &= \int_{-\infty}^{\infty} x^{2n} |x|^{2\lambda+1} \exp\left(-x^4 + tx^2\right) dx \\ &= \mu_0(t;\lambda+n) = \frac{d^n}{dt^n}\mu_0(t,\lambda), \quad n = 1, 2, \ldots\end{aligned}$$

whilst the odd ones are

$$\mu_{2n+1}(t;\lambda) = \int_{-\infty}^{\infty} x^{2n+1} |x|^{2\lambda+1} \exp\left(-x^4 + tx^2\right) dx = 0, \quad n = 1, 2, \ldots$$

since the integrand is odd.

The recurrence coefficients

Monic orthogonal polynomials with respect to the generalized Freud weight

$$|x|^{2\lambda+1}\exp(-x^4+tx^2)$$

satisfy the three-term recurrence relation

$$xS_n(x;t) = S_{n+1}(x;t) + \beta_n(t;\lambda)S_{n-1}(x;t)$$

where $\beta_n(t;\lambda) > 0$, $S_{-1}(x;t) = 0$, $S_0(x;t) = 1$, $\beta_0(t;\lambda) = 0$
and

$$\beta_1(t;\lambda) = \frac{\mu_2(t;\lambda)}{\mu_0(t;\lambda)} = \frac{\int_{-\infty}^{\infty} x^2|x|^{2\lambda+1}\exp\left(-x^4+tx^2\right)dx}{\int_{-\infty}^{\infty}|x|^{2\lambda+1}\exp\left(-x^4+tx^2\right)dx}$$

$$= \tfrac{1}{2}t + \tfrac{1}{2}\sqrt{2}\,\frac{D_{-\lambda}\left(-\tfrac{1}{2}\sqrt{2}\,t\right)}{D_{-\lambda-1}\left(-\tfrac{1}{2}\sqrt{2}\,t\right)},$$

...

The differential-difference equation

The coefficients $A_n(x;t)$ and $B_n(x;t)$ in the relation

$$\frac{dP_n}{dx}(x;t) = -B_n(x;t)P_n(x;t) + A_n(x;t)P_{n-1}(x;t), \tag{1}$$

satisfied by semi-classical orthogonal polynomials are of interest since differentiation of this differential-difference equation yields the second order differential equation satisfied by the orthogonal polynomials.

- Shohat [1939] gave a procedure using quasi-orthogonality to derive (1) for weights $w(x;t)$ such that $w'(x;t)/w(x;t)$ is a rational function;
- This technique was rediscovered by several authors including Bonan, Freud, Mhaskar and Nevai approximately 40 years later;
- Method of ladder operators was introduced by Chen and Ismail [1997];
- Chen and Feigin [2006] adapt the method of ladder operators to the situation where the weight function vanishes at one point.
- Clarkson, J and Kelil [2016] generalize the work by Chen and Feigin, giving a more explicit expression for the coefficients in (1) when the weight function is positive on the real line except for one point.

The derivatives of monic orthogonal polynomials $S_n(x;t)$ with respect to the generalized Freud weight are quasi-orthogonal of order $m = 5$.

$$x\frac{dS_n}{dx}(x;t) = \sum_{k=n-4}^{n} c_{n,k}S_k(x;t),$$

where the coefficient $c_{n,k}$, for $n-4 \leq k \leq n$ and $h_k > 0$, is given by

$$c_{n,k} = \frac{1}{h_k}\int_{-\infty}^{\infty}\frac{dS_n(x;t)}{dx}xS_k(x;t)w(x;t)\,dx.$$

Integrating by parts, we obtain for $n-4 \leq k \leq n-1$,

$$\begin{aligned}
h_k c_{n,k} &= \Big[xS_k(x;t)S_n(x;t)w(x;t)\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{d}{dx}\left[xS_k(x;t)w(x;t)\right]S_n(x;t)\,dx \\
&= -\int_{-\infty}^{\infty}\left[S_n(x;t)S_k(x;t) + xS_n(x;t)\frac{dS_k(x;t)}{dx}\right]w(x;t)\,dx \\
&\qquad - \int_{-\infty}^{\infty}xS_n(x;t)S_k(x;t)\frac{dw(x;t)}{dx}\,dx \\
&= -\int_{-\infty}^{\infty}xS_n(x;t)S_k(x;t)\frac{dw(x;t)}{dx}\,dx \\
&= -\int_{-\infty}^{\infty}S_n(x;t)S_k(x;t)\left(-4x^4 + 2tx^2 + 2\lambda + 1\right)w(x;t)\,dx \\
&= \int_{-\infty}^{\infty}(4x^4 - 2tx^2)S_n(x;t)S_k(x;t)w(x;t)\,dx.
\end{aligned}$$

Iterating the three-term recurrence relation, the following relations are obtained:

$$\begin{aligned} x^2 S_n &= S_{n+2} + (\beta_n + \beta_{n+1}) S_n + \beta_n \beta_{n-1} S_{n-2}, \\ x^4 S_n &= S_{n+4} + (\beta_n + \beta_{n+1} + \beta_{n+2} + \beta_{n+3}) S_{n+2} \\ &\quad + \left[\beta_n(\beta_{n-1} + \beta_n + \beta_{n+1}) + \beta_{n+1}(\beta_n + \beta_{n+1} + \beta_{n+2})\right] S_n \\ &\quad + \beta_n \beta_{n-1}(\beta_{n-2} + \beta_{n-1} + \beta_n + \beta_{n+1}) S_{n-2} \\ &\quad + (\beta_n \beta_{n-1} \beta_{n-2} \beta_{n-3}) S_{n-4}. \end{aligned}$$

Substituting these into

$$h_k c_{n,k} = \int_{-\infty}^{\infty} (4x^4 - 2tx^2) S_n(x;t) S_k(x;t) w(x;t)\, dx$$

yields the coefficients $\{c_{n,k}\}_{k=n-4}^{n-1}$ in

$$x \frac{d S_n}{dx}(x;t) = \sum_{k=n-4}^{n} c_{n,k} S_k(x;t),$$

as follows:

$$\begin{aligned} c_{n,n-4} &= 4\beta_n \beta_{n-1} \beta_{n-2} \beta_{n-3}, \\ c_{n,n-3} &= 0, \\ c_{n,n-2} &= 4\beta_n \beta_{n-1}(\beta_{n-2} + \beta_{n-1} + \beta_n + \beta_{n+1} - \tfrac{1}{2} t), \\ c_{n,n-1} &= 0. \end{aligned}$$

Lastly, we consider the case when $k = n$.

A similar (but longer) argument and the fact that the recurrence coefficient β_n satisfies discrete $\mathrm{P_I}$

$$\beta_{n+1} + \beta_n + \beta_{n-1} = \tfrac{1}{2}t + \frac{2n + (2\lambda + 1)[1 - (-1)^n]}{8\beta_n}$$

yields

$$c_{n,n} = n.$$

Now we can write

$$x\frac{dS_n}{dx}(x;t) = c_{n,n-4}S_{n-4}(x;t) + c_{n,n-2}S_{n-2}(x;t) + c_{n,n}S_n(x;t). \qquad (2)$$

In order to express S_{n-4} and S_{n-2} in terms of S_n and S_{n-1}, we iterate the recurrence relation.

The differential-difference equation

Theorem (Clarkson, J & Kelil, 2016)

For the generalized Freud weight

$$w(x;t) = |x|^{2\lambda+1} \exp\left(-x^4 + tx^2\right), \qquad x \in \mathbb{R},\ \lambda > 0$$

the monic orthogonal polynomials $S_n(x;t)$ satisfy

$$\frac{d S_n}{dx}(x;t) = -B_n(x;t)S_n(x;t) + A_n(x;t)S_{n-1}(x;t)$$

with

$$A_n(x;t) = 4\beta_n(x^2 - \tfrac{1}{2}t + \beta_n + \beta_{n+1}),$$
$$B_n(x;t) = 4x\beta_n + \frac{(2\lambda+1)[1-(-1)^n]}{2x}.$$

The differential equation

Theorem (Clarkson, J & Kelil, 2016)

For the generalized Freud weight, the monic orthogonal polynomials $S_n(x;t)$ *satisfy the differential equation*

$$\frac{d^2 S_n}{dx^2}(x;t) + R_n(x;t)\frac{d\,S_n}{dx}(x;t) + T_n(x;t)S_n(x;t) = 0,$$

$$R_n(x;t) = -4x^3 + 2tx + \frac{2\lambda+1}{x} - \frac{2x}{x^2 - \frac{1}{2}t + \beta_n + \beta_{n+1}},$$

$$\begin{aligned} T_n(x;t) = {} & 4nx^2 + 4\beta_n + 16\beta_n(\beta_n + \beta_{n+1} - \tfrac{1}{2}t)(\beta_n + \beta_{n-1} - \tfrac{1}{2}t) \\ & + 4(2\lambda+1)(-1)^n\beta_n - \frac{8\beta_n x^2 + (2\lambda+1)[1-(-1)^n]}{x^2 - \frac{1}{2}t + \beta_n + \beta_{n+1}} \\ & + (2\lambda+1)[1-(-1)^n]\left(t - \frac{1}{2x^2}\right). \end{aligned}$$

Note that the coefficients in the differential equation of the semiclassical orthogonal polynomials are not the same as those of the Pearson equation.