14th OPSFA

University of Kent

3-7 July 2017

The 14th International Symposium on
Orthogonal Polynomials and Special Functions and Applications

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Jonathan Breuer (Hebrew University of Jerusalem, Israel)

Title. **To infinity and back (a bit)**

Abstract. Let H be a self-adjoint operator defined on an infinite dimensional Hilbert space. Given some spectral information about H, such as the continuity of its spectral measure, what can be said about the asymptotic spectral properties of its finite dimensional approximations? This is a natural (and general) question, and can be used to frame many specific problems such as the asymptotics of zeros of orthogonal polynomials, or eigenvalues of random matrices. We shall discuss some old and new results in the context of this general framework and present various open problems.

Sylvie Corteel (CNRS, Paris, France)

Title. **Koornwinder polynomials at** $q = t$

Abstract. In this talk, I will explain how to build Koornwinder polynomials at q=t from moments of Askey Wilson polynomials. I will use the classical combinatorial theory of Viennot for orthogonal polynomials and their moments. An extension of this theory allows to build multivariate orthogonal polynomials. The key step for this construction are a Cauchy identity for Koornwinder polynomials due to Mimachi and a Jacobi-Trudi formula for the 9th variation of Schur functions due to Nakagawa, Noumi, Shirakawa and Yamada. This is joint work with Olya Mandelshtam and Lauren Williams.

David Gomez-Ullate (ICMAT and Universidad Complutense de Madrid, Spain)

Title. **Exceptional orthogonal polynomials**

Abstract. Exceptional orthogonal polynomials came as a little surprise to the community a few years ago, and now we are beginning to understand large parts of the theory and how they fit within the existing framework. They are complete families of orthogonal polynomials that arise as eigenfunctions of a Sturm-Liouville problem, and generalize the classical families of Hermite, Laguerre and Jacobi by allowing a finite number of gaps in their degree sequence. Despite these "missing" degrees, the remaining polynomials still span a complete basis of a weighted L2 space, and the orthogonality weight is a rational modification of a classical weight. In this talk we will review the main results in the theory of exceptional

orthogonal polynomials (classification, position of their zeros, recurrence relations, etc.), with emphasis on the similarities and differences with respect to their classical counterparts. We will also discuss some of their applications in mathematical physics, ranging from exact solutions to Schrodinger's equation in Quantum Mechanics to rational solutions of nonlinear integrable equations of Painlevé type.

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Evelyne Hubert (INRIA, Sophia Antipolis, France)

Title. Computing Symmetric cubatures: A moment matrix approach.

Abstract. A quadrature is an approximation of the definite integral of a function by a weighted sum of function values at specified points, or nodes, within the domain of integration. Gaussian quadratures are constructed to yield exact results for any polynomials of degree $2r-1$ or less by a suitable choice of r nodes and weights. Cubature is a generalization of quadrature in higher dimension. Constructing a cubature amounts to find a linear form
$\Lambda : \mathbb{R}[x] \to \mathbb{R}$, $p \mapsto \sum_{j=1}^{r} a_j\, p(\xi_j)$ from the knowledge of its restriction to $\mathbb{R}[x]_{\leq d}$. The unknowns to be determined are the weights a_j and the nodes ξ_j.

An approach based on moment matrices was proposed in [4, 6, 2]. We give a basis-free version in terms of the Hankel operator $\mathcal{H}$ associated to Λ. The existence of a cubature of degree d with r nodes boils down to conditions of ranks and positive semidefiniteness on $\mathcal{H}$. We then recognize the nodes as the solutions of a generalized eigenvalue problem.

Standard domains of integration are symmetric under the action of a finite group. It is natural to look for cubatures that respect this symmetry [3, 5]. They are exact for all anti-symmetric functions beyond the degree of the cubature. Introducing adapted bases obtained from representation theory, the symmetry constraint allows to block diagonalize the Hankel operator $\mathcal{H}$. The size of the blocks is explicitly related to the orbit types of the nodes. From the computational point of view, we then deal with smaller-sized matrices both for securing the existence of the cubature and computing the nodes.

Joint work with Mathieu Collowald, Université Côte d'Azur & Inria Méditerranée [1].

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Arieh Iserles (University of Cambridge, UK)

Title. From Fourier's sombrero to Chebyshev's almond: approximation on the real line

Abstract. Motivated by the computation of quantum problems, we explore fast approximation of functions on the real line, in particular of wave packets – by "fast" we mean both spectral speed of convergence and derivation of the first n expansion coefficients in $O(n \log n)$ operations. We explore four candidates: Hermite polynomials, Hermite functions, stretched Fourier series and stretched Chebyshev series, describe some unexpected phenomena and determine the surprising winner.

Alexander Its (Indiana University-Purdue University, Indianapolis, USA)

Title. Monodromy dependence and connection formulae for isomonodromic tau functions

Abstract. We discuss an extension of the Jimbo-Miwa-Ueno differential 1-form to a form closed on the full space of extended monodromy data of systems of linear ordinary differential equation with rational coefficients. This extension is based on the results of M. Bertola generalizing a previous construction by B. Malgrange. We show how this 1-form can be used to solve a long-standing problem of evaluation of the connection formulae for the generic isomonodromic tau functions which would include an explicit computation of the relevant constant factors. We explain how this scheme works by calculating the connection constants for generic Painlevé VI and Painlevé III tau functions. The result proves the conjectural formulae for these constants earlier proposed by N. Iorgov, O. Lisovyy, and Yu. Tykhyy (PVI) and by O. Lisovyy, Y. Tykhyy, and the speaker (PIII) with the help of the recently discovered connection of the Painlevé tau-functions with the Virasoro conformal blocks. The conformal block approach will be also outlined. The talk is based on the joint works with O. Lisovyy, Y. Tykhyy and A. Prokhorov.

Arno Kuijlaars (KU Leuven, Belgium)

Title. Universality for conditional measures of the sine point process

Abstract. The sine process is a random point process that is obtained as a limit from the eigenvalues of many random matrices as the size tends to infinity. This phenomenon is called universality in random matrix theory, and it also holds for many orthogonal polynomial ensembles.

In this talk I want to emphasize another connection of the sine point process with orthogonal polynomials. It comes from a surprising property called number rigidty in the sense of Ghosh and Peres. This means that for almost all configurations, the number of points in an interval [-R,R] is determined exactly by the points outside the interval. The conditional measures is the joint distribution of the points in [-R,R] given the points outside. Bufetov showed that these are orthogonal polynomial ensembles with a weight that comes from the points outside [-R,R].

I will report on recent work with Erwin Mina-Diaz (arXiv:1703.02349) where we prove a universality result for these orthogonal polynomial ensembles that in particular implies that the correlation kernel

of the orthogonal polynomial ensemble tends to the sine kernel as R tends to infinity. This answers a question posed by Alexander Bufetov.

Marta Mazzocco (Loughborough University, UK)

Title. **Painlevé equations, q-Askey scheme and colliding holes on Riemann surfaces**

Abstract. The monodromy manifold of the sixth Painlevé differential equation is a Poisson algebra that admits a natural quantisation to the Askey-Wilson algebra, encoding the symmetries of the Askey-Wilson polynomials. On the other side, this same monodromy manifold is the moduli space of monodromy representations of a Riemann sphere with 4 boundary components. In this talk we will show that by merging boundary components on this Riemann sphere, other Painlevé equations emerge in such a way that the quantization of their monodromy manifolds encode the symmetries of other families of basic hypergeometric polynomials belonging to the q-Askey scheme.

Peter Miller (University of Michigan, Ann Arbor, USA)

Title. **Rational Solutions of Painlevé Equations**

Abstract. Most solutions of the famous Painlevé differential equations are highly transcendental, yet all but the Painlevé-I equation admit particular solutions that are elementary rational functions. These particular solutions are important in diverse applications, including the description of equilibrium patterns of fluid vortices, universal phenomena in nonlinear wave theory, electrochemistry, and string theory. This talk will illustrate some features of these particular solutions, describe how they may be obtained by iterated Bäcklund transformations or via the solution of appropriate Riemann-Hilbert problems, and focus attention on recent and ongoing efforts by several researchers to study families of rational Painlevé solutions in the asymptotic limit of large degree.

Margit Rösler (University of Paderborn, Germany)

Title. **Integral representations for multivariable Bessel functions and beta distributions**

Abstract. There exist various interesting classes of multivariable Bessel functions, such as Bessel functions of matrix argument which are important in multivariate statistics, or the Bessel functions associated with root systems in Dunkl theory. Such Bessel functions occur for certain discrete parameters in various contexts of radial analysis on Euclidean spaces.

In this talk, we shall first review some basics on Bessel functions of matrix argument and Dunkl-type Bessel functions, and explain their interrelation. We shall then focus on integral representations for multivariable Bessel functions which generalize the classical Sonine integral for the one-variable Bessel function $j_\alpha(z) = {}_0F_1(\alpha+1; -z^2/4)$,

$$j_{\alpha+\beta}(z) = \frac{2\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta)} \int_0^1 j_\alpha(zt)\, t^{2\alpha+1}(1-t^2)^{\beta-1}dt \quad (\alpha > -1, \beta > 0).$$

For Bessel functions of matrix argument there are analogous representations, going back already to Herz, and for certain Bessel functions associated with root systems of type B there are similar results by Macdonald. In these known cases, however, the range of indices is restricted. Similar to the theory of Gindikin for Riesz measures, we shall extend these integral representations to larger index sets by means of tempered distributions, und study under which conditions these distributions are actually given by positive measures. There turn out to be gaps in the admissible range of indices which are determined by the so-called Wallach set.

As a consequence, we shall obtain examples where the Dunkl intertwining operator between Dunkl operators associated with multiplicities $k \geq 0$ and $k' \geq k$ is not positive, which disproves a long-standing conjecture.

The talk is based on joint work with Michael Voit, Dortmund.

Nina Snaith (University of Bristol, UK)

Title. **Every moment brings a treasure: random matrix theory and moments of the Riemann zeta function**

Abstract. There has been very convincing evidence since the 1970s that the positions of zeros of the Riemann zeta function show the same statistical distribution (in the appropriate limit) as eigenvalues of

random matrices. This talk will review how this connection was exploited in order to gain insight into average values of the zeta function and its derivative.

Jacek Szmigielski (University of Saskatchewan, Saskatoon, Canada)

Title. Non-smooth waves and Lax integrability; the playground of approximation theory and the theory of distributions

Abstract. In the last two decades several models have been proposed to describe non-smooth waves with integrable structure, the best known of which are the Camassa-Holm (CH) and Degasperis-Processi (DP) equations and their numerous generalizations. These equations possess stable, non-smooth solutions, called peakons, which, to a large extent, determine the essential properties of solutions, in particular the breakdown of regularity and the onset of shocks (DP). The non-smooth character of these solutions presents a considerable challenge for the question of Lax integrability since the underlying idea of commutativity of partial derivatives no longer holds and one is forced to use ideas of distribution theory mindful that distributional operations require a special care for non-linear problems. In this talk, I will review the pertinent inverse boundary value problems coming from distributional boundary value problems relevant for the peakon sectors of several of these "peakon" equations, with the due emphasis on the approximation theory aspects which play a decisive role in the con- struction of these solutions. Some of the highlights of this connection involve Stieltljes continued fractions, Hermite-Padé approximations, multipoint Padé approximations, oscillatory kernels of Gantmacher-Krein type and Nikishin systems. In the second half of my talk I will concentrate on my recent work with Xiangke Chang (Beijing) on the modi

ed Camassa-Holm equation for which many of the challenges posed by non-smoothness are present.

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Tom Trogdon (University of California, Irvine, USA) – Gabor Szegő prize winner 2017

Title. **The high oscillation of special functions**

Abstract. High oscillation in mathematics is ubiquitous and essential. Answering asymptotic questions often relies critically on tools, such as the method of steepest descent, to capture oscillation. In this talk, I will discuss settings where high oscillation governs the fundamental features of a problem but presents intrinsic asymptotic and/or numerical barriers. Examples include short-time behavior of dispersive PDEs with discontinuous initial data and causal Wiener filters with delay. Special functions, and their related techniques, are often the key to overcoming oscillation-induced difficulties.