Parameter Redundancy and Identifiability

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Heron Example

- Heron (*Ardea cinerea*) ‘census’ data

$$\boldsymbol{y} = \begin{bmatrix} 5100 \\ 5075 \\ 5125 \\ \vdots \end{bmatrix}$$

- State-space model (Besbeas *et al,* 2002)

$$\begin{bmatrix} N_{1,t} \\ N_{2,t} \\ N_{a,t} \end{bmatrix} = \begin{bmatrix} 0 & \rho\phi_1 & \rho\phi_1 \\ \phi_2 & 0 & 0 \\ 0 & \phi_a & \phi_a \end{bmatrix} \begin{bmatrix} N_{1,t-1} \\ N_{2,t-1} \\ N_{a,t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \\ \epsilon_{a,t} \end{bmatrix}$$

$$y_t = [0 \quad 1 \quad 1] \begin{bmatrix} N_{1,t} \\ N_{2,t} \\ N_{a,t} \end{bmatrix} + \eta_t$$

$N_{i,t}$ - number of individuals in ith age class at time t;
ϕ_i survival probabilities; ρ productivity; η_t and ϵ_i error processes.

- The two parameters ρ and ϕ_1 only ever appear as $\rho\phi_1$. It will only ever be possible to estimate the product and never the two parameters separately.
- This is an example of parameter redundancy or non-identifiability.
- Can we estimate ϕ_2 and ϕ_a?

Detecting Parameter Redundancy / Identifiability

- Symbolic method (Cole *et al*, 2010, Cole and McCrea, 2016)
- Form a vector, $\boldsymbol{\kappa}$, representing model:

$$\boldsymbol{A} = \begin{bmatrix} 0 & \rho\phi_1 & \rho\phi_1 \\ \phi_2 & 0 & 0 \\ 0 & \phi_a & \phi_a \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}, \boldsymbol{c}_0 = \begin{bmatrix} N_{1,1} \\ N_{2,1} \\ N_{a,1} \end{bmatrix}$$

$$\boldsymbol{y}_z = \boldsymbol{B}(z\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{A}\boldsymbol{c}_0 = \frac{-(\phi_2 N_{1,1} + \phi_a N_{2,1} + \phi_a N_{a,1})z - \rho\phi_1\phi_2(N_{2,1} + N_{a,1})}{-z^2 + \phi_a z + \rho\phi_1\phi_2}$$

$$\boldsymbol{\kappa} = \begin{bmatrix} -\rho\phi_1\phi_2(N_{2,1} + N_{a,1}) \\ -(\phi_2 N_{1,1} + \phi_a N_{2,1} + \phi_a N_{a,1}) \\ \rho\phi_1\phi_2 \\ \phi_a \end{bmatrix}$$

Detecting Parameter Redundancy / Identifiability

$$\boldsymbol{\kappa} = \begin{bmatrix} -\rho\phi_1\phi_2(N_{2,1} + N_{a,1}) \\ -(\phi_2 N_{1,1} + \phi_a N_{2,1} + \phi_a N_{a,1}) \\ \rho\phi_1\phi_2 \\ \phi_a \end{bmatrix}$$

- Form a derivative matrix

$$\boldsymbol{\theta} = [\phi_1 \quad \phi_2 \quad \phi_a \quad \rho]$$

$$\boldsymbol{D} = \frac{\partial\boldsymbol{\kappa}}{\partial\boldsymbol{\theta}} = \begin{bmatrix} -\rho\phi_2(N_{2,1} + N_{a,1}) & 0 & \rho\phi_2 & 0 \\ -\rho\phi_1\,(N_{2,1} + N_{a,1}) & -N_{1,1} & \rho\phi_1 & 0 \\ 0 & -N_{2,1} - N_{a,1} & 0 & 1 \\ -\phi_1\phi_2(N_{2,1} + N_{a,1}) & 0 & \phi_1\phi_2 & 0 \end{bmatrix}$$

- Rank($\boldsymbol{D}$) = 3. There are 4 parameters.
- If rank is less than number of parameters, the model is parameter redundant / non-identifiable.
- Solving PDEs (derived from ***D***) we can estimate $\phi_1\rho, \phi_2, \phi_a$.

Detecting Parameter Redundancy / Identifiability

- Symbolic method extends to more complex problems where identifiability is less obvious. (Some examples in Cole and McCrea, 2016).
- Are there other complex SSM where methods are useful?
- Other methods exist to detect parameter identifiability, but can lead to the wrong conclusion. For example Abadi *et al* (2010) wrongly state can estimate immigration parameter without separate data on productivity.
- Other there other examples of this? What are the consequences (e.g. bias estimates)?
- Symbolic and other methods to detect identifiability discussed on Thursday morning.

References

- Abadi, F., Gimenez, O**.,** Arlettaz, R. and Schaub, M. (2010). An assessment of integrated population models: bias, accuracy, and violation of the assumption of independence. *Ecology,* **91**, 7-14.
- Besbeas, P., Freeman, S. N., Morgan, B. J. T. and Catchpole, E. A. (2002) Integrating Mark-Recapture-Recovery and Census Data to Estimate Animal Abundance and Demographic Parameters. *Biometrics*, **58**, 540-547.
- Cole, D. J., Morgan, B. J. T and Titterington, D. M. (2010) Determining the Parametric Structure of Non-Linear Models. *Mathematical Biosciences*, **228**, 16-30.
- Cole, D. J. and McCrea, R. M. (2016). Parameter Redundancy in Discrete State-Space and Integrated Models. *Biometrical Journal*, 58, 1071-1090.